

PROBLEMS OF HARMONIC ANALYSIS RELATED TO FINITE TYPE HYPERSURFACES IN \mathbb{R}^3 , AND NEWTON POLYHEDRA

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ABSTRACT. This article is intended to give an overview on a collection of results which have been obtained jointly with I.I. Ikromov, and in parts also with M. Kempe, and at the same time to give a kind of guided tour through the rather comprehensive proofs of the major results that I shall address. All of our work is highly influenced by the pioneering ideas developed by E.M. Stein.

Consider a smooth hypersurface S in \mathbb{R}^3 of finite type, and let $d\mu = \rho d\sigma$ be a surface carried measure with smooth, compactly supported density $\rho \geq 0$ with respect to the surface measure $d\sigma$. The problems that I shall address are the following ones:

A. Find, if possible, optimal uniform decay estimates for the Fourier transform of the surface carried measure $d\mu$.

B. If we denote by A_t the averaging operator $A_t f(x) := \int_S f(x - ty) d\mu(y)$, determine for which exponents p the associated maximal operator

$$\mathcal{M}f(x) := \sup_{t>0} |A_t f(x)|$$

is bounded on $L^p(\mathbb{R}^3)$.

C. Determine the range of exponents p for which a Fourier restriction estimate

$$\left(\int_S |\hat{f}(x)|^2 d\mu(x) \right)^{1/2} \leq C \|f\|_{L^p(\mathbb{R}^3)}$$

holds true. The first problem is classical, and the other ones originate from seminal contributions by E.M. Stein. These problems are somewhat interrelated, and in particular it is well-known that the answers to B and C are strongly influenced by the answer to A. Moreover, due to the insights by V.I. Arnol'd and his school, in particular through work by A.N. Varchenko and V.N. Karpushkin, we know that the answer to problem A can be given in terms of Newton polyhedra associated to the given hypersurface S , at least for analytic hypersurfaces. That Newton polyhedra would play an important role also in other problems of harmonic analysis has been stressed later in pioneering work by D.H. Phong and E.M. Stein on oscillatory integral operators.

In this article, I shall outline how (almost) complete answers to the above questions can be given again in terms of associated Newton polyhedra (for question B, at least if $p > 2$).

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1. INTRODUCTION

Let S be a smooth, finite type hypersurface in \mathbb{R}^3 with Riemannian surface measure $d\sigma$, and consider the compactly supported measure $d\mu := \rho d\sigma$ on S , where $0 \leq \rho \in C_0^\infty(S)$.

The problems on which I shall essentially focus are the following ones:

A. Find, if possible, optimal uniform decay estimates for the Fourier transform of the surface carried measure $d\mu$.

B. If we denote by A_t the averaging operator $A_t f(x) := \int_S f(x - ty) d\mu(y)$, determine for which exponents p the associated maximal operator

$$\mathcal{M}f(x) := \sup_{t>0} |A_t f(x)|$$

is bounded on $L^p(\mathbb{R}^3)$. For instance, if S is a Euclidean sphere centered at the origin, then \mathcal{M} is the spherical maximal operator studied first by Stein [61].

C. Determine the range of exponents p for which a Fourier restriction estimate

$$\left(\int_S |\hat{f}(x)|^2 d\mu(x) \right)^{1/2} \leq C \|f\|_{L^p(\mathbb{R}^3)}$$

holds true. I shall explain how these questions can be answered in an (almost) complete way in terms of Newton polyhedra associated to the given surface S . All these results

are based on joint work with Ikromov, and in parts also with Kempe [39], [40], [37], [38] [41],[42].

Problem A is a classical question about estimates for oscillatory integrals, and there exists a huge body of results on it, in particular for convex hypersurfaces (some references, also to higher dimensional results, will follow).

The other two problems had first been formulated by Stein: the study of maximal averages along hypersurfaces has been initiated in Stein's work on the spherical maximal function [61], and also the idea of Fourier restriction goes back to him (cf. §5). Their great importance for the study of partial differential equations became clear through Strichartz' article [63], and in the PDE-literature their dual versions are often called Strichartz estimates.

Of course, these and related problems, such as the question of optimal sub-level estimates and integrability indices, make perfect sense also in higher dimensions (and possibly higher co-dimension), but only partial answers are known to most of these problems in this case. The reason for this is that strong information on the resolution of singularities is available for analytic functions of two variables, for instance by means of Puiseux series expansions of roots, whereas the situation for multivariate functions of more than two variables is substantially more complex. Nevertheless, there has been a lot of progress on the question as to how to construct more elementary and "concrete" resolutions of singularities for real analytic multivariate functions, giving more detailed information than what Hironaka's celebrated theorem [35] on the resolution of singularities would yield, for instance in work by Bierstone and P.D. Milman [7], [8], Sussman [64], Parusiński [53], [54], Greenblatt [28], [30], Collins, Greenleaf and Pramanik [15], among others. These techniques have already led to a very good understanding of, for instance, the sublevel estimation problem for slices of the surfaces in direction to the Gaussian normal, and the related determination of critical integrability indices, in independent work by Greenblatt [28], [30], and also Collins, Greenleaf and Pramanik [15], by rather different methods. Yet another approach has been developed by Denef, Nicaise and Sargos [21] in order to estimate oscillatory integrals in higher dimensions.

These encouraging results give rise to the hope that eventually, substantial progress might also be possible in higher dimensions for the deeper problems A - C, but at present, only partial answers are available, for restricted classes of surfaces (see, e.g., [31], [33]). What makes these problems indeed a lot harder than the sub-level set problem is that it will surely require estimates of oscillatory integrals with phase functions depending also on small parameters, not just a fixed phase function. This leads to the fundamental problem of stability of the estimates in Problem A under small perturbation; I shall briefly come back to this later.

Before returning to the two-dimensional case, let me address some results on the three problems A - C which have been obtained for particular classes of hypersurfaces also in higher dimensions.

The case of convex hypersurface of finite line type has been studied quite intensively, for instance in work by Randol [57], Svensson [65], Schulz [59], Bruna, Nagel and Wainger [12] and Cowling, Disney, Mauceri and myself [17] concerning problem A, and for instance by Nagel, Seeger and Wainger [52] and Iosevich, Sawyer and Seeger [46] on problem B. Various particular classes of non-convex hypersurfaces have been examined too, for instance in work by Iosevich and Sawyer [44], Cowling and Mauceri [18], [19], and by Sogge and Stein [60], where the method of damping (with powers of the Gaussian curvature) had been introduced in order to derive partial answers to problem B for very general hypersurfaces in \mathbb{R}^n .

However, from now on I shall concentrate on the two-dimensional case, and give only occasionally a few references to work in higher dimensions. Since all three problems can be seen as "classical" by now, there is an abundance of literature associated to them, so that it would seem impossible to give due credit to everyone who has made contributions, and I apologize in advance to everyone whose work is not mentioned, due to lack of space or my personal ignorance. Many further references can be found in the cited articles, and for a more detailed account of the state of the theory and its historical development until roughly a decade ago, I refer, for instance, to Stein's monograph [62].

It is obvious that all three problems A - C can be localized to sufficiently small neighborhoods of given points x^0 on S . Observe also that the problems A and B are invariant under translations and rotations of the ambient space, so that we may replace the surface S by any suitable image under a Euclidean motion of \mathbb{R}^3 .

We may thus assume that $x^0 = (0, 0, 0)$, and that S is the graph

$$(1.1) \quad S = \{(x_1, x_2, \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\},$$

of a smooth function ϕ defined on a sufficiently small neighborhood Ω of the origin, such that

$$(1.2) \quad \phi(0, 0) = 0, \quad \nabla \phi(0, 0) = 0.$$

The situation is different for problem B, since dilations do not commute with translations, so that we are only allowed to work with linear transformations of the ambient space. We shall therefore study the maximal operator \mathcal{M} under the following transversality assumption on S .

Assumption 1.1. *The affine tangent plane $x + T_x S$ to S through x does not pass through the origin in \mathbb{R}^3 for every $x \in S$. Equivalently, $x \notin T_x S$ for every $x \in S$, so that $0 \notin S$, and x is transversal to S for every point $x \in S$.*

Notice that this assumption allows us to find a linear change of coordinates in \mathbb{R}^3 so that in the new coordinates S can locally be represented as the shifted graph of a function ϕ as before, more precisely,

$$(1.3) \quad S = \{(x_1, x_2, 1 + \phi(x_1, x_2)) : (x_1, x_2) \in \Omega\},$$

where ϕ satisfies again (1.2).

Our transversality assumption is natural in this context. Indeed, various examples show that if it is not satisfied, then the behavior of the corresponding maximal function may change drastically.

Observe also that if ϕ is flat, i.e., if all derivatives of ϕ vanish at the origin, and if $\rho(x^0) > 0$, then it is well-known and easy to see that the maximal operator \mathcal{M} is L^p -bounded if and only if $p = \infty$, so that this case is of no interest. In a similar way, also the problems A and C will become of quite a different nature. We shall therefore assume that ϕ is non-flat, i.e., of finite type. Correspondingly, we shall always assume that the hypersurface S is of finite type, in the sense that every tangent plane has finite order of contact. Recall that in the study a convex hypersurface, a standard assumption used by many authors is that the surface is of finite line type, which means that every tangent line has finite order of contact, which is stronger than our assumption.

The article will be organized as follows:

In the next paragraph, I shall briefly review some basic concepts concerning Newton polyhedra, adaptedness of coordinates and the notion of height, which have been introduced by Arnol'd (cf. [2], [3]) and his school, most notably Varchenko [66]. These concepts had originally been studied for real analytic functions ϕ , but as shown in [39], can be extended to smooth, finite type functions ϕ . Problems A to C will then be discussed subsequently in §3 – §5.

Since many of our proofs are quite elaborate, my main goal will be to give a kind of guided tour through the basic structure of the proofs, hoping that this might also be helpful for a everyone interested in studying some of the proofs in more detail.

2. NEWTON POLYHEDRA, AND ADAPTED COORDINATES

Let me first recall some basic notions from [66],[39]. If ϕ is given as before, consider the associated Taylor series

$$\phi(x_1, x_2) \sim \sum_{\alpha_1, \alpha_2=0}^{\infty} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$$

of ϕ centered at the origin. The set

$$\mathcal{T}(\phi) := \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 : c_{\alpha_1, \alpha_2} = \frac{1}{\alpha_1! \alpha_2!} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \phi(0, 0) \neq 0\}$$

will be called the *Taylor support* of ϕ at $(0, 0)$. We shall always assume that

$$\mathcal{T}(\phi) \neq \emptyset,$$

i.e., that the function ϕ is of finite type at the origin. The *Newton polyhedron* $\mathcal{N}(\phi)$ of ϕ at the origin is defined to be the convex hull of the union of all the quadrants $(\alpha_1, \alpha_2) + \mathbb{R}_+^2$ in \mathbb{R}^2 , with $(\alpha_1, \alpha_2) \in \mathcal{T}(\phi)$. The associated *Newton diagram* $\mathcal{N}_d(\phi)$ in the sense of Varchenko [66] is the union of all compact faces of the Newton polyhedron; here, by a *face*, we shall mean an edge or a vertex.

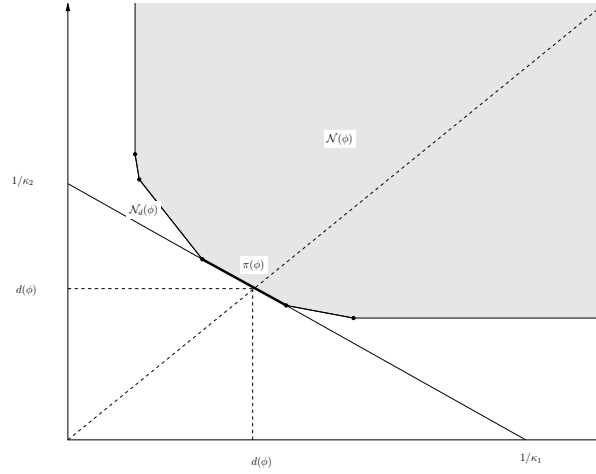


FIGURE 1. Newton polyhedron

We shall use coordinates (t_1, t_2) for points in the plane containing the Newton polyhedron, in order to distinguish this plane from the (x_1, x_2) - plane.

The *Newton distance*, or shorter *distance* $d = d(\phi)$ between the Newton polyhedron and the origin in the sense of Varchenko is given by the coordinate d of the point (d, d) at which the bi-sectrix $t_1 = t_2$ intersects the boundary of the Newton polyhedron.

The *principal face* $\pi(\phi)$ of the Newton polyhedron of ϕ is the face of minimal dimension containing the point (d, d) . Deviating from the notation in [66], we shall call the series

$$\phi_{\text{pr}}(x_1, x_2) := \sum_{(\alpha_1, \alpha_2) \in \pi(\phi)} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$$

the *principal part* of ϕ . In case that $\pi(\phi)$ is compact, ϕ_{pr} is a mixed homogeneous polynomial; otherwise, we shall consider ϕ_{pr} as a formal power series.

Note that the distance between the Newton polyhedron and the origin depends on the chosen local coordinate system in which ϕ is expressed. By a *local coordinate system at the origin* I shall mean a smooth coordinate system defined near the origin which preserves 0. The *height* of the smooth function ϕ is defined by

$$h(\phi) := \sup\{d_y\},$$

where the supremum is taken over all local coordinate systems $y = (y_1, y_2)$ at the origin, and where d_y is the distance between the Newton polyhedron and the origin in the coordinates y .

A given coordinate system x is said to be *adapted* to ϕ if $h(\phi) = d_x$.

In [39] we proved that one can always find an adapted local coordinate system in two dimensions, thus generalizing the fundamental work by Varchenko [66] who worked in the setting of real-analytic functions ϕ (see also [56]).

Recall also that if the principal face of the Newton polyhedron $\mathcal{N}(\phi)$ is a compact edge, then it lies on a unique “principal line”

$$L := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\},$$

with $\kappa_1, \kappa_2 > 0$. By permuting the coordinates x_1 and x_2 , if necessary, we shall always assume that $\kappa_1 \leq \kappa_2$. The weight $\kappa = (\kappa_1, \kappa_2)$ will be called the *principal weight* associated to ϕ . It induces dilations $\delta_r(x_1, x_2) := (r^{\kappa_1} x_1, r^{\kappa_2} x_2)$, $r > 0$, on \mathbb{R}^2 , so that the principal part ϕ_{pr} of ϕ is κ -homogeneous of degree one with respect to these dilations, i.e., $\phi_{\text{pr}}(\delta_r(x_1, x_2)) = r \phi_{\text{pr}}(x_1, x_2)$ for every $r > 0$, and

$$(2.1) \quad d = \frac{1}{\kappa_1 + \kappa_2} = \frac{1}{|\kappa|}.$$

It can then easily be shown (cf. Proposition 2.2 in [39]) that ϕ_{pr} can be factorized as

$$(2.2) \quad \phi_{\text{pr}}(x_1, x_2) = c x_1^{\nu_1} x_2^{\nu_2} \prod_{l=1}^M (x_2^q - \lambda_l x_1^p)^{n_l},$$

with $M \geq 1$, distinct non-trivial “roots” $\lambda_l \in \mathbb{C} \setminus \{0\}$ of multiplicities $n_l \in \mathbb{N} \setminus \{0\}$, and trivial roots of multiplicities $\nu_1, \nu_2 \in \mathbb{N}$ at the coordinate axes. Here, p and q have no common divisor, and $\kappa_2/\kappa_1 = p/q$.

More generally, if $\kappa = (\kappa_1, \kappa_2)$ is any weight with $0 < \kappa_1 \leq \kappa_2$ such that the line $L_\kappa := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1 t_1 + \kappa_2 t_2 = 1\}$ is a supporting line to the Newton polyhedron $\mathcal{N}(\phi)$ of ϕ , then the κ -principal part of ϕ

$$\phi_\kappa(x_1, x_2) := \sum_{(\alpha_1, \alpha_2) \in L_\kappa} c_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}$$

is a non-trivial polynomial which is κ -homogeneous of degree 1 with respect to the dilations associated to this weight as before, and which can be factorized in a similar way as in (2.2). By definition, we then have

$$(2.3) \quad \phi(x_1, x_2) = \phi_\kappa(x_1, x_2) + \text{terms of higher } \kappa\text{-degree}.$$

Adaptedness of a given coordinate system can be verified by means of the following criterion (see [39]): Denote by

$$m(\phi_{\text{pr}}) := \text{ord}_{S^1} \phi_{\text{pr}}$$

the maximal order of vanishing of ϕ_{pr} along the unit circle S^1 centered at the origin. The *homogeneous distance* of a κ -homogeneous polynomial P (such as $P = \phi_{\text{pr}}$) is given by $d_h(P) := 1/(\kappa_1 + \kappa_2) = 1/|\kappa|$. Notice that $(d_h(P), d_h(P))$ is just the point of intersection of the line given by $\kappa_1 t_1 + \kappa_2 t_2 = 1$ with the bi-sectrix $t_1 = t_2$. The height of P can then be computed by means of the formula

$$(2.4) \quad h(P) = \max\{m(P), d_h(P)\}.$$

In [39] (Corollary 4.3 and Corollary 2.3), we proved the following characterization of adaptedness of a given coordinate system:

Proposition 2.1. *The coordinates x are adapted to ϕ if and only if one of the following conditions is satisfied:*

- (a) *The principal face $\pi(\phi)$ of the Newton polyhedron is a compact edge, and $m(\phi_{\text{pr}}) \leq d(\phi)$.*
- (b) *$\pi(\phi)$ is a vertex.*
- (c) *$\pi(\phi)$ is an unbounded edge.*

These conditions had already been introduced by Varchenko, who has shown that they are sufficient for adaptedness when ϕ is analytic.

We also note that in case (a) we have $h(\phi) = h(\phi_{\text{pr}}) = d_h(\phi_{\text{pr}})$. Moreover, it can be shown that (a) applies whenever $\pi(\phi)$ is a compact edge and $\kappa_2/\kappa_1 \notin \mathbb{N}$; in this case we even have $m(\phi_{\text{pr}}) < d(\phi)$ (cf. [39], Corollary 2.3).

2.1. Construction of adapted coordinates. In the case where the coordinates (x_1, x_2) are not adapted to ϕ , the previous results show that the principal face $\pi(\phi)$ must be a compact edge, that $m := \kappa_2/\kappa_1 \in \mathbb{N}$, and that $m(\phi_{\text{pr}}) > d(\phi)$. One easily verifies that this implies that $p = m, q = 1$ in (2.2), and that there is at least one, non-trivial real root $x_2 = \lambda_l x_1$ of ϕ_{pr} of multiplicity $n_l = m(\phi_{\text{pr}}) > d(\phi)$. Indeed, one can show that this root is unique. Putting $b_1 := \lambda_l$, we shall denote the corresponding root $x_2 = b_1 x_1$ of ϕ_{pr} as its *principal root*.

Changing coordinates

$$y_1 := x_1, \quad y_2 := x_2 - b_1 x_1^m,$$

we arrive at a “better” coordinate system $y = (y_1, y_2)$. Indeed, this change of coordinates will transform ϕ_{pr} into a function $\widetilde{\phi}_{\text{pr}}$, where the principal face of $\widetilde{\phi}_{\text{pr}}$ will be a horizontal half-line at level $t_2 = m(\phi_{\text{pr}})$, so that $d(\widetilde{\phi}_{\text{pr}}) > d(\phi)$, and correspondingly one finds that $d(\widetilde{\phi}) > d(\phi)$, if $\widetilde{\phi}$ expresses ϕ in the coordinates y (cf. [39]).

Somewhat oversimplifying, by iterating this procedure, we essentially arrive at Varchenko’s algorithm for the construction of an adapted coordinate system (cf. [39] for details).

In conclusion, one can show that there exists a smooth real-valued function ψ (which we may choose as the so-called *principal root jet* of ϕ) of the form

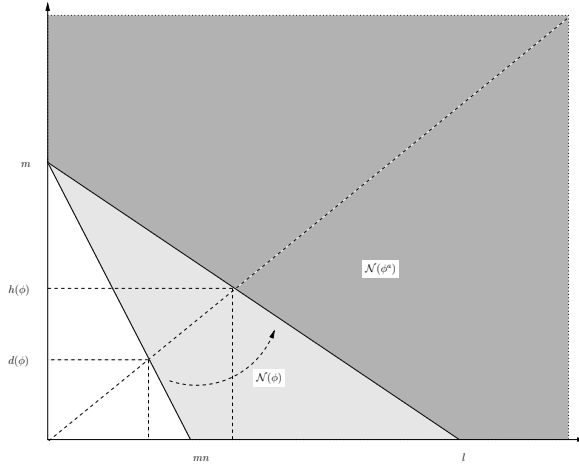
$$(2.5) \quad \psi(x_1) = c x_1^m + O(x_1^{m+1})$$

with $c \neq 0$, defined on a neighborhood of the origin such that an adapted coordinate system (y_1, y_2) for ϕ is given locally near the origin by means of the (in general non-linear) shear

$$(2.6) \quad y_1 := x_1, \quad y_2 := x_2 - \psi(x_1).$$

In these adapted coordinates, ϕ is given by

$$(2.7) \quad \phi^a(y) := \phi(y_1, y_2 + \psi(y_1)).$$

FIGURE 2. $\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell$ ($\ell > mn$)**Example 2.2.**

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n + x_1^\ell.$$

Assume that $\ell > mn$. Then the coordinates are not adapted. Indeed, $\phi_{\text{pr}}(x_1, x_2) = (x_2 - x_1^m)^n$, $d(\phi) = 1/(1/n + 1/(mn)) = mn/(m+1)$ and $m(\phi_{\text{pr}}) = n > d(\phi)$. Adapted coordinates are given by $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is expressed by $\phi^a(y) = y_2^n + y_1^\ell$.

Remark 2.3. *An alternative proof of Varchenko's theorem on the existence of adapted coordinates for analytic functions ϕ of two variables has been given by Phong, J. Sturm and Stein in [56], by means of Puisseux series expansions of the roots of ϕ .*

Let us finally observe that when $m = \kappa_2/\kappa_1 = 1$ in the first step of Varchenko's algorithm, then a linear change of coordinates of the form $y_1 = x_1, y_2 = x_2 - b_1 x_1$ will transform ϕ into a function $\tilde{\phi}$. Since all of our problems A - C are invariant under such linear changes of coordinates, by replacing our original coordinates (x_1, x_2) by (y_1, y_2) and ϕ by $\tilde{\phi}$, we may in the sequel always assume without loss of generality that either our coordinates (x_1, x_2) are adapted, or they are not adapted and

$$(2.8) \quad m = \kappa_2/\kappa_1 \quad \text{is an integer} \quad \geq 2.$$

A linear, non-adapted coordinate system for which (2.8) holds true will be called *linearly adapted* to ϕ .

3. PROBLEM A: DECAY OF THE FOURIER TRANSFORM OF THE SURFACE CARRIED MEASURE $d\mu$.

Observe first that in view of (1.1), we may write $\hat{\mu}(\xi)$ as an oscillatory integral

$$\hat{\mu}(\xi) =: J(\xi) = \int_{\Omega} e^{-i(\xi_3 \phi(x_1, x_2) + \xi_1 x_1 + \xi_2 x_2)} \eta(x) dx, \quad \xi \in \mathbb{R}^3,$$

where $\eta \in C_0^\infty(\Omega)$. Since $\nabla\phi(0,0) = 0$, the complete phase in this oscillatory integral will have no critical point on the support of η unless $|\xi_1| + |\xi_2| \ll |\xi_3|$, provided Ω is chosen sufficiently small. Integrations by parts then show that $\hat{\mu}(\xi) = O(|\xi|^{-N})$ as $|\xi| \rightarrow \infty$, for every $N \in \mathbb{N}$, unless $|\xi_1| + |\xi_2| \ll |\xi_3|$.

We may thus focus on the latter case. In this case, by writing $\lambda = -\xi_3$ and $\xi_j = s_j\lambda$, $j = 1, 2$, we are reduced to estimating two-dimensional oscillatory integrals of the form

$$I(\lambda; s) := \int e^{i\lambda(\phi(x_1, x_2) + s_1x_1 + s_2x_2)} \eta(x_1, x_2) dx_1 dx_2,$$

where we may assume without loss of generality that $\lambda \gg 1$, and that $s = (s_1, s_2) \in \mathbb{R}^2$ is sufficiently small, provided that η is supported in a sufficiently small neighborhood of the origin. The complete phase function is thus a small, linear perturbation of the function ϕ .

If $s = 0$, then the function $I(\lambda; 0)$ is given by an oscillatory integral of the form $\int e^{i\lambda\phi(x)} \eta(x) dx$, and it is well-known ([6], [4]) that for any analytic phase function ϕ defined on a neighborhood of the origin in \mathbb{R}^n such that $\phi(0) = 0$, such an integral admits an asymptotic expansion as $\lambda \rightarrow \infty$ of the form

$$(3.1) \quad \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} a_{j,k}(\phi) \lambda^{-r_k} \log(\lambda)^j,$$

provided the support of η is sufficiently small. Here, the r_k form an increasing sequence of rational numbers consisting of a finite number of arithmetic progressions, which depends only on the zero set of ϕ , and the $a_{j,k}$ are distributions with respect to the cut-off function η . The proof is based on Hironaka's theorem.

Let us come back to the case $n = 2$. Following [66] (with a slight modification), we next define what we like to call *Varchenko's exponent* $\nu(\phi) \in \{0, 1\}$:

If there exists an adapted local coordinate system y near the origin such that the principal face $\pi(\phi^a)$ of ϕ , when expressed by the function ϕ^a in the new coordinates, is a vertex, and if $h(\phi) \geq 2$, then we put $\nu(\phi) := 1$; otherwise, we put $\nu(\phi) := 0$. We remark [40] that the first condition is equivalent to the following one: *If y is any adapted local coordinate system at the origin, then either $\pi(\phi^a)$ is a vertex, or a compact edge and $m(\phi_{\text{pr}}^a) = d(\phi^a)$.*

Varchenko [66] has shown that the leading exponent in (3.1) is given by $r_0 = 1/h(\phi)$, and $\nu(\phi)$ is the maximal j for which $a_{j,k}(\phi) \neq 0$. This implies in particular that

$$(3.2) \quad |I(\lambda; 0)| \leq C \lambda^{-\frac{1}{h(\phi)}} \log(\lambda)^{\nu(\phi)}, \quad \lambda \gg 1,$$

and this estimate is sharp in the exponents. Subsequently, Karpushkin [47] proved that this estimate is stable under sufficiently small analytic perturbations of ϕ (analogous results are known to be wrong in higher dimensions [66]). In particular, we find that $J(\lambda; s)$ satisfies the same estimate (3.2) for $|s|$ sufficiently small, so that we obtain the following *uniform estimate* for $\hat{\mu}$,

$$(3.3) \quad |\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\frac{1}{h(\phi)}} \log(2 + |\xi|)^{\nu(\phi)}, \quad \xi \in \mathbb{R}^3,$$

provided the support of ρ is sufficiently small.

In [40], we proved, by a quite different method, that Karpushkin's result remains valid for smooth, finite type functions ϕ , at least for linear perturbations, which led to the following

Theorem 3.1. *Let $S = \text{graph}(\phi)$ be as before, and assume that ϕ is smooth and of finite type. Then estimate (3.3) holds true provided the support of ρ is sufficiently small.*

The special case where $\xi = (0, 0, \xi_3)$ is normal to S at the origin is due to Greenblatt [29].

One can also show that this estimate is sharp in the exponents even when ϕ is not analytic, except for the case where the principal face $\pi(\phi^a)$ is an unbounded edge. Indeed, if $\pi(\phi^a)$ is compact, then [40]

$$I(\lambda; 0) \asymp C \lambda^{-\frac{1}{h(\phi)}} \log(\lambda)^{\nu(\phi)}$$

as $\lambda \rightarrow +\infty$, where C is a non-zero constant. However, when $\pi(\phi^a)$ is unbounded then the following examples, due to Iosevich and Sawyer [45], show that there may be a different behavior in general: Let $\phi(x_1, x_2) := x_2^2 + e^{-1/|x_1|^\alpha}$, with $\alpha > 0$; then

$$|I(\lambda; 0)| \asymp \frac{1}{\lambda^{1/2} \log \lambda^{1/\alpha}} \text{ as } \lambda \rightarrow +\infty,$$

whereas $\nu(\phi) = 0$.

It appears likely that even the full analog of Karpushkin's theorem, i.e., stability of estimate (3.2) under arbitrary small, smooth perturbations, holds true in two dimensions.

3.1. Outline of some main ideas of the proof. Our proof of Theorem 3.1 is based on a certain decomposition scheme for the given surface S , related to the Newton polyhedra of ϕ respectively ϕ^a , which has been inspired by the work of Phong and Stein, in combination with re-scaling arguments. Since the same type of decompositions plays an important role also in the more involved proofs of our results on the other problems B and C (which require also further, more subtle refinements of them), I shall outline some of the major ideas used in the proof subsequently.

We first notice that the case where $h(\phi) < 2$ is covered by Duistermaat's work [24] (notice that Duistermaat proves estimates of the form (3.3) without the presence of a logarithmic factor $\log(2 + |\xi|)$, even for a wider class of phase functions). Note that the required estimates can also be derived easily from the normal forms in Theorem 5.8.

I shall therefore subsequently assume that

$$h := h(\phi) \geq 2.$$

In many situations, one can reduce the problem to a one-dimensional one by means of van der Corput's lemma, respectively the following (not quite straight-forward) consequence of it, whose formulation goes back to J. E. Björk (see [22]) and G. I. Arhipov [1].

Lemma 3.2. *Assume that f is a smooth real valued function defined on an interval $I \subset \mathbb{R}$ which is of polynomial type $n \geq 2$ ($n \in \mathbb{N}$), i.e., there are positive constants $c_1, c_2 > 0$ such that*

$$c_1 \leq \sum_{j=2}^n |f^{(j)}(s)| \leq c_2 \quad \text{for every } s \in I.$$

Then for $\lambda \in \mathbb{R}$,

$$\left| \int_I e^{i\lambda f(s)} g(s) ds \right| \leq C \|g\|_{C^1(I)} (1 + |\lambda|)^{-1/n},$$

where the constant C depends only on the constants c_1 and c_2 .

3.1.1. *The case where the coordinates are adapted to ϕ .* Let us write $d = d(\phi)$, and recall that here $h = d$, since the coordinates are adapted. Assume for instance that the principal face $\pi(\phi)$ is a compact edge. By decomposing \mathbb{R}^2 into the half-spaces $\mathbb{R}_\pm^2 := \mathbb{R} \times \mathbb{R}_\pm$, we may also assume that the integration in $J(\xi)$ takes place over one of these half-spaces only, say, \mathbb{R}_+^2 . Let κ be the principal weight, with associated dilations $\delta_r(x_1, x_2) = (r^{\kappa_1} x_1, r^{\kappa_2} x_2)$. Recall also that then $\phi_{\text{pr}} = \phi_\kappa$ is δ_r -homogeneous of degree 1.

We fix a suitable smooth cut-off function χ on \mathbb{R}^2 supported in an annulus \mathcal{A} on which $|x| \sim 1$, such that the functions $\chi_k := \chi \circ \delta_{2^k}$ form a partition of unity, and then decompose

$$J(\xi) = \sum_{k=k_0}^{\infty} J_k(\xi),$$

where

$$J_k(\xi) := \int_{\mathbb{R}_+^2} e^{-i(\xi_3 \phi(x) + \xi_1 x_1 + \xi_2 x_2)} \eta(x) \chi_k(x) dx.$$

Scaling by $\delta_{2^{-k}}$, we see that

$$(3.4) \quad J_k(\xi) = 2^{-k|\kappa|} \int_{\mathbb{R}_+^2} e^{-i\left(2^{-k}\xi_3\phi^k(x) + 2^{-k\kappa_1}\xi_1x_1 + 2^{-k\kappa_2}\xi_2x_2\right)} \eta(\delta_{2^{-k}}(x)) \chi(x) dx,$$

with $\phi^k(x) := 2^k \phi(\delta_{2^{-k}} x)$. Notice that in view of (2.3),

$$\phi^k(x) = \phi_\kappa(x) + \text{error term}.$$

We claim that given any point $x^0 \in \mathcal{A}$, we can find a unit vector $e \in \mathbb{R}^2$ and some $j \in \mathbb{N}$ with $2 \leq j \leq h$ such that $\partial_e^j \phi_\kappa(x^0) \neq 0$, where ∂_e denotes the partial derivative in direction of e .

Indeed, if $\nabla\phi_\kappa(x^0) \neq 0$, then the homogeneity of ϕ_κ and Euler's homogeneity relation imply that $\text{rank}(D^2\phi_\kappa(x^0)) \geq 1$. Therefore, we can find a unit vector $e \in \mathbb{R}^2$ such that $\partial_e^2\phi_\kappa(x^0) \neq 0$. And, if $\nabla\phi_\kappa(x^0) = 0$, then by Euler's homogeneity relation we have $\phi_\kappa(x^0) = 0$ as well. Thus the function ϕ_κ vanishes in x^0 of order $j \geq 2$. But, in view of Proposition 2.1, then $j \leq m(\phi_{\text{pr}}) \leq d = h$, which verifies the claim.

For $k \geq k_0$ sufficiently large we can thus apply van der Corput's lemma to the integration along lines parallel to the direction e in the integral defining $J_k(\xi)$ near the point x^0 . Applying Fubini's theorem and a partition of unity argument, we thus obtain

$$(3.5) \quad \begin{aligned} |J_k(\xi)| &\leq C\|\eta\|_{C^3(\mathbb{R}^2)} 2^{-k|\kappa|} (1 + 2^{-k}|\xi_3|)^{-1/j} \\ &\leq C\|\eta\|_{C^3(\mathbb{R}^2)} 2^{-k|\kappa|} (1 + 2^{-k}|\xi|)^{-1/M}, \end{aligned}$$

where M denotes the maximal j that arises in this context.

Summation in k then yields the following estimates:

$$(3.6) \quad |J(\xi)| \leq C\|\eta\|_{C^3(\mathbb{R}^2)} \begin{cases} (1 + |\xi|)^{-1/M}, & \text{if } M|\kappa| > 1, \\ (1 + |\xi|)^{-1/M} \log(2 + |\xi|), & \text{if } M|\kappa| = 1, \\ (1 + |\xi|)^{-|\kappa|}, & \text{if } M|\kappa| < 1. \end{cases}$$

However, since we are assuming that $\pi(\phi)$ is a compact edge, we have that $1/|\kappa| = d(\phi) = h$, and moreover $M \leq h$. This implies $|\kappa|M \leq 1$. Since we have seen that here $\nu(\phi) = 1$ if and only if $M = m(\phi_{\text{pr}}) = h$, i.e., if and only if $M|\kappa| = 1$, we obtain estimate (3.3).

3.1.2. The case where the coordinates are not adapted to ϕ . Here, in a *first step*, we may reduce to a narrow neighborhood of the principal root.

Indeed, away from the principal root of ϕ_{pr} , we can argue in the same way as before, since the multiplicity of any real root of ϕ_{pr} different from the principal root is bounded by $d \leq h$. I.e., we can reduce to a narrow κ -homogeneous neighborhood of the curve $x_2 = b_1 x_1^m$, of the form

$$(3.7) \quad |x_2 - b_1 x_1^m| \leq \varepsilon x_1^m,$$

say by means of a function $\rho_1(x) := \chi_0((x_2 - b_1 x_1^m)/(\varepsilon x_1^m))$, where χ_0 is a suitable smooth bump function supported in the interval $[-1, 1]$ and $\varepsilon > 0$ is sufficiently small. I.e., in place of $J(\xi)$, it suffices to estimate $J^{\rho_1}(\xi)$, where we write

$$J^\chi(\xi) := \int_{\mathbb{R}_+^2} e^{-i(\xi_3\phi(x_1, x_2) + \xi_1 x_1 + \xi_2 x_2)} \eta(x) \chi(x) dx$$

if χ is any integrable function.

Second Step: Domain decomposition into "homogeneous" domains D_l and transition domains E_l . In order to study the contribution $J^{\rho_1}(\xi)$ by the domain (3.7), we change to the adapted coordinates y and essentially re-write

$$(3.8) \quad J^{\rho_1}(\xi) = \int_{\mathbb{R}_+^2} e^{-i(\xi_3\phi^a(y_1, y_2) + \xi_1 y_1 + \xi_2 \psi(y_1) + \xi_2 y_2)} \tilde{\eta}(y) \tilde{\chi}_0\left(\frac{y_2}{\varepsilon y_1^m}\right) dy,$$

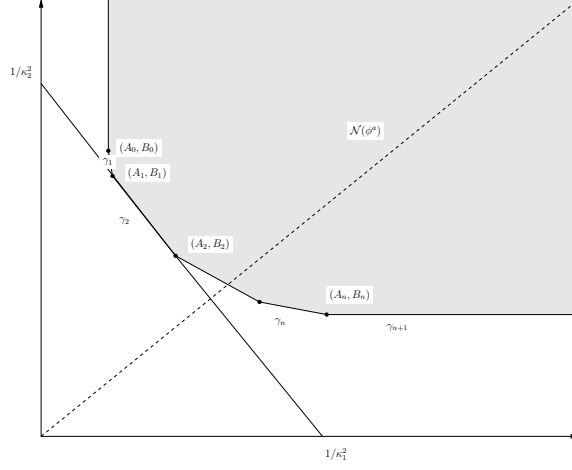


FIGURE 3. Edges and weights

where $\tilde{\eta}$ and $\tilde{\chi}_0$ have properties similar to η , respectively χ_0 .

Let us denote the vertices of the Newton polyhedron $\mathcal{N}(\phi^a)$ by (A_l, B_l) , $l = 0, \dots, n$, where we assume that they are ordered so that $A_{l-1} < A_l$, $l = 1, \dots, n$, with associated compact edges given by the intervals $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)]$, $l = 1, \dots, n$. The unbounded horizontal edge with left endpoint (A_n, B_n) will be denoted by γ_{n+1} . To each of these edges γ_l , we associate the weight $\kappa^l = (\kappa_1^l, \kappa_2^l)$, so that γ_l is contained in the line

$$L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^l t_1 + \kappa_2^l t_2 = 1\}.$$

For $l = n + 1$, we have $\kappa_1^{n+1} := 0, \kappa_2^{n+1} = 1/B_n$. We denote by

$$a_l := \frac{\kappa_2^l}{\kappa_1^l}, \quad l = 1, \dots, n$$

the reciprocal of the slope of the line L_l . For $l = n + 1$, we formally set $a_{n+1} := \infty$.

If $l \leq n$, the κ^l -principal part $\phi_{\kappa^l}^a$ of ϕ^a corresponding to the supporting line L_l is of the form

$$(3.9) \quad \phi_{\kappa^l}^a(y) = c_l y_1^{A_{l-1}} y_2^{B_l} \prod_{\alpha} (y_2 - c_l^{\alpha} y_1^{a_l})^{N_{\alpha}}$$

(cf. [38]).

Remark 3.3. When ϕ is analytic, then this expression is linked to the Puiseux series expansion of roots of ϕ as follows [55] (compare also [39]):

We may then factorize

$$\phi^a(y_1, y_2) = U(y_1, y_2) y_1^{\nu_1} y_2^{\nu_2} \prod_r (y_2 - r(y_1)),$$

where the product is indexed by all non-trivial roots $r = r(y_1)$ of ϕ^a (which may also be empty) and where $U(0, 0) \neq 0$. Moreover, these roots can be expressed in a small

neighborhood of 0 as Puiseux series

$$r(y_1) = c_{l_1}^{\alpha_1} y_1^{a_{l_1}} + c_{l_1 l_2}^{\alpha_1 \alpha_2} y_1^{a_{l_1 l_2}^{\alpha_1}} + \cdots + c_{l_1 \dots l_p}^{\alpha_1 \dots \alpha_p} y_1^{a_{l_1 \dots l_p}^{\alpha_1 \dots \alpha_{p-1}}} + \cdots,$$

where

$$c_{l_1 \dots l_p}^{\alpha_1 \dots \alpha_{p-1} \beta} \neq c_{l_1 \dots l_p}^{\alpha_1 \dots \alpha_{p-1} \gamma} \quad \text{for } \beta \neq \gamma,$$

$$a_{l_1 \dots l_p}^{\alpha_1 \dots \alpha_{p-1}} > a_{l_1 \dots l_{p-1}}^{\alpha_1 \dots \alpha_{p-2}},$$

with strictly positive exponents $a_{l_1 \dots l_p}^{\alpha_1 \dots \alpha_{p-1}} > 0$ and non-zero complex coefficients $c_{l_1 \dots l_p}^{\alpha_1 \dots \alpha_p} \neq 0$, and where we have kept enough terms to distinguish between all the non-identical roots of ϕ^a . The leading exponents in these series are the numbers

$$a_1 < a_2 < \cdots < a_n.$$

One can therefore group the roots into the clusters of roots $[l], l = 1, \dots, n$, where the l 'th cluster $[l]$ consists of all roots with leading exponent a_l .

Correspondingly, we can decompose

$$\phi^a(y_1, y_2) = U(y_1, y_2) y_1^{\nu_1} y_2^{\nu_2} \prod_{l=1}^n \Phi_{[l]}(y_1, y_2),$$

where

$$\Phi_{[l]}(y_1, y_2) := \prod_{r \in [l]} (y_2 - r(y_1)).$$

Observe the following: If $\delta_s^l(x_1, x_2) = (s^{\kappa_1^l} x_1, s^{\kappa_2^l} x_2)$, $s > 0$, denote the dilations associated to the weight κ^l , and if $r \in [l_1]$ is a root in the cluster $[l_1]$, then one easily checks that for $y = (y_1, y_2)$ in a bounded set we have $\delta_s^l y_2 = s^{\kappa_2^l} y_2$ and $r(\delta_s^l y_1) = s^{a_{l_1} \kappa_1^l} c_{l_1}^{\alpha_1} y_1^{a_{l_1}} (1 + O(s^\varepsilon))$ as $s \rightarrow 0$, for some $\varepsilon > 0$. Consequently,

$$\delta_s^l y_2 - r(\delta_s^l y_1) = (1 + O(s^\varepsilon)) \begin{cases} -s^{a_{l_1} \kappa_1^l} c_{l_1}^{\alpha_1} y_1^{a_{l_1}}, & \text{if } l_1 < l, \\ s^{\kappa_2^l} (y_2 - c_l^{\alpha_l} y_1^{a_l}), & \text{if } l_1 = l, \\ s^{\kappa_2^l} y_2, & \text{if } l_1 > l. \end{cases}$$

This shows that the κ^l -principal part of ϕ^a is given by

$$(3.10) \quad \phi_{\kappa^l}^a = C_l y_1^{\nu_1 + \sum_{l_1 < l} |[l_1]| a_{l_1}} y_2^{\nu_2 + \sum_{l_1 > l} |[l_1]|} \prod_{\alpha_1} (y_2 - c_l^{\alpha_l} y_1^{a_l})^{N_{l, \alpha_1}},$$

where N_{l, α_1} denotes the number of roots in the cluster $[l]$ with leading term $c_l^{\alpha_l} y_1^{a_l}$. A look at the Newton polyhedron reveals that the exponents of y_1 and y_2 in (3.10) can be expressed in terms of the vertices (A_j, B_j) of the Newton polyhedron:

$$\nu_1 + \sum_{l_1 < l} |[l_1]| a_{l_1} = A_{l-1}, \quad \nu_2 + \sum_{l_1 > l} |[l_1]| = B_l.$$

Notice also that

$$\prod_{\alpha_1} (y_2 - c_l^{\alpha_1} y_1^{a_l})^{N_{l,\alpha_1}} = \prod_{l=1}^n (\Phi_{[l]})_{\kappa^l}.$$

Comparing this with (3.9), the close relation between the Newton polyhedron of ϕ^a and the Pusieux series expansion of roots becomes evident, and accordingly we say that the edge $\gamma_l := [(A_{l-1}, B_{l-1}), (A_l, B_l)]$ is *associated to the cluster of roots* $[l]$.

Next we choose the integer $l_0 \geq 1$ such that

$$a_1 < \dots < a_{l_0-1} \leq m < a_{l_0} < \dots < a_l < a_{l+1} < \dots < a_n.$$

Since the original coordinates x were assumed to be non-adapted, the vertex (A_{l_0-1}, B_{l_0-1}) will lie strictly above the bisectrix, i.e., $A_{l_0-1} < B_{l_0-1}$.

Let us consider the case where the principal face of the Newton polyhedron of ϕ^a is a **compact edge** (the other cases require modified arguments). We choose $\lambda > l_0$ so that the edge $\gamma_\lambda = [(A_{\lambda-1}, B_{\lambda-1}), (A_\lambda, B_\lambda)]$ is the principal face $\pi(\phi^a)$ of the Newton polyhedron of ϕ^a (cf. Figure 3, where $\lambda = 3$.) In the case where $m(\phi_{\text{pr}}^a) = d(\phi^a)$, it can easily be seen that by running Varchenko's algorithm one step further, we can pass to new adapted coordinates in which the principal face is a vertex.

We therefore may assume that $m(\phi_{\text{pr}}^a) < d(\phi^a)$, so that $\nu(\phi) = 0$.

We shall narrow down the domain (3.7) to a neighborhood D_λ of the principal root jet of the form

$$(3.11) \quad |x_2 - \psi(x_1)| \leq N_\lambda x_1^{a_\lambda},$$

where N_λ is a constant to be chosen later. This domain is κ^λ -homogeneous in the adapted coordinates y .

To this end, following an idea from [55], we decompose the difference set of the domains (3.7) and (3.11) into the domains

$$D_l := \{(x_1, x_2) : \varepsilon_l x_1^{a_l} < |x_2 - \psi(x_1)| \leq N_l x_1^{a_l}\}, \quad l = l_0, \dots, \lambda - 1,$$

and the intermediate domains

$$E_l := \{(x_1, x_2) : N_{l+1} x_1^{a_{l+1}} < |x_2 - \psi(x_1)| \leq \varepsilon_l x_1^{a_l}\}, \quad l = l_0, \dots, \lambda - 1,$$

as well as $E_{l_0-1} := \{(x_1, x_2) : N_{l_0} x_1^{a_{l_0}} < |x_2 - \psi(x_1)| \leq \varepsilon_1 x_1^{a_1}\}$. Here, the $\varepsilon_l > 0$ are small and the $N_l > 0$ are large parameters to be chosen suitably.

Observe that the domain

$$D_l^a := \{(y_1, y_2) : \varepsilon_l y_1^{a_l} < |y_2| \leq N_l y_1^{a_l}\}$$

corresponding to D_l is κ^l -homogeneous in the adapted coordinates y given by (2.2), and contains the cluster of roots $[l]$, if ε_l and N_l are chosen sufficiently small, respectively large, while the domain E_l^a corresponding to E_l can be viewed as a domain of transition between two different homogeneities.

The contributions by the domain D_l to $J^{\rho_1}(\xi)$ can again be estimated in a similar way as we did in the adapted case, by using dyadic decompositions and subsequent re-scalings by means of the dilations δ_r^l associated to the weight κ^l .

More precisely, the corresponding k 'th term will be given by the following analogue of (3.4)

$$J_k(\xi) = 2^{-k|\kappa^l|} \int_{\mathbb{R}_+^2} e^{-i\left(2^{-k}\xi_3\phi^k(y) + \xi_2\psi(2^{-k\kappa_1^l}y_1) + 2^{-k\kappa_1^l}\xi_1y_1 + 2^{-k\kappa_2^l}\xi_2y_2\right)} \eta(\delta_{2^{-k}}^l y) \chi(y) dy,$$

where $\phi^k(y) = \phi_{\kappa^l}^a(y) + \text{error term}$.

Notice, however, that since $1 - m\kappa_1^l > \kappa_2^l - m\kappa_1^l > 0$, the contribution of the non-linearity ψ to the complete phase of the corresponding oscillatory integrals may be large, compared to the term containing ϕ^k , so that we are only allowed to apply van der Corput's estimate to the integration with respect to the variable y_2 if we want to reduce to one-dimensional oscillatory integrals!

We therefore need a *control on the multiplicities of roots of $\partial_2^2\phi_{\kappa^l}^a$* at points y^0 in the corresponding annulus \mathcal{A} not lying on the y_1 axis (note that the latter corresponds to the principal root jet in the coordinates y). Indeed, it can be shown (cf. Proposition 2.3 (b) in [38]) that these multiplicities are bounded by $d_h(\phi_{\kappa^l}^a) - 2$, where $d_h(\phi_{\kappa^l}^a)$ denotes the homogeneous distance of $\phi_{\kappa^l}^a$, and it is evident from the geometry of the Newton polyhedron of ϕ^a that $d_h(\phi_{\kappa^l}^a) < d(\phi^a) = h$, so that for every point y^0 in $\mathcal{A} \cap D_l$ there is some $j \in \{2, \dots, h-1\}$ such that

$$\partial_2^j \phi_{\kappa^l}^a(y^0) \neq 0.$$

As for the contributions by the domains E_l , here we perform a separate dyadic decomposition in both variables y_1 and y_2 , so that we geometrically decompose E_l into dyadic rectangles of size $2^{-j} \times 2^{-k}$, and then re-scale in both variables so that these rectangles become the standard cube, say, $[1, 2] \times [1, 2]$.

Now, the estimates of Section 11 in [38] show that the phase functions $\phi_{j,k}^a$ that one obtains after these re-scalings satisfy the estimate

$$\partial_2^2 \phi_{j,k}^a(y^0) \neq 0 \quad \text{for every } y^0 \in [1, 2] \times [1, 2].$$

Since $h \geq 2$, this clearly suffices to obtain the necessary order of decay of the Fourier transform of these dyadic pieces. Moreover, scaling back to the original dyadic rectangles, a careful analysis of the dependency of the corresponding estimates on the parameters j, k shows that it is indeed possible to sum these estimates and obtain the same type of estimate for the contributions by the domains E_l as for the domains D_l , even without logarithmic factor.

Note that so far, we have always been able to reduce our estimations to van der Corput's lemma, respectively, Lemma 3.2, i.e., to one-dimensional oscillatory integrals.

3.1.3. *Third Step: Study of the contribution by the homogenous domain D_λ containing the principal root jet.* What remains to be estimated is the contribution of the domain (3.11) to $J(\xi)$. As we shall see, the study of this domain will in certain cases require the estimation of genuinely 2-dimensional oscillatory integrals, and a reduction to the one-dimensional case is no longer possible.

We recall that according to our convention

$$(3.12) \quad m(\phi_{\text{pr}}^a) < d(\phi^a) = h,$$

so that $\nu(\phi) = 0$. In the adapted coordinates y , the domain (3.11) is given by $|y_2| \leq N_\lambda y_1^{a_\lambda}$, and we can cover it by a finite number of κ^λ -homogeneous subdomains of the form $|y_2 - cy_1^{a_\lambda}| \leq \varepsilon_0 y_1^{a_\lambda}$, where $c \in [-N_\lambda, N_\lambda]$, and where, for a given c , we may choose $\varepsilon_0 > 0$ suitably small.

Writing $\psi(x_1) = x_1^m \omega(x_1)$, with a smooth function ω satisfying $\omega(0) \neq 0$, we can thus reduce to estimating oscillatory integrals of the form

$$(3.13) \quad J^c(\xi) = \int_{\mathbb{R}_+^2} e^{iF(y, \xi)} \rho\left(\frac{y_2 - cy_1^{a_\lambda}}{\varepsilon_0 y_1^{a_\lambda}}\right) \eta(y) dy,$$

with a phase function

$$F(y, \xi) := \xi_3 \phi^a(y) + \xi_1 y_1 + \xi_2 y_1^m \omega(y_1) + \xi_2 y_2$$

depending on $\xi \in \mathbb{R}^3$.

Arguing in a similar way as in the case of adapted coordinates, and recalling that $\phi_{\text{pr}}^a = \phi_{\kappa^\lambda}^a$, we may again perform a dyadic decomposition and re-scale by means of the dilations δ_r^λ , in order to write

$$J^c(\xi) = \sum_{k=k_0}^{\infty} J_k(\xi),$$

where

$$(3.14) \quad J_k(\xi) = 2^{-|\kappa^\lambda|k} \int e^{i2^{-k} \xi_3 F_k(y, s)} \rho\left(\frac{y_2 - cy_1^{a_\lambda}}{\varepsilon_0 y_1^{a_\lambda}}\right) \eta(\delta_{2^{-k}}^\lambda y) \chi(y) dy,$$

where

$$F_k(y, s) := \phi_{\text{pr}}^a(y_1, y_2) + s_1 y_1 + S_2 y_1^m \omega(2^{-\kappa_1^\lambda k} y_1) + s_2 y_2 + \text{error},$$

where $s := (s_1, s_2, S_2)$ is given by

$$s_1 := 2^{(1-\kappa_1^\lambda)k} \frac{\xi_1}{\xi_3}, \quad s_2 := 2^{(1-\kappa_2^\lambda)k} \frac{\xi_2}{\xi_3}, \quad S_2 := 2^{(\kappa_2^\lambda - m\kappa_1^\lambda)k} s_2.$$

Note that $2 \leq m < a_\lambda = \kappa_2^\lambda / \kappa_1^\lambda$ and $k \gg 1$, so that $|S_2| \gg |s_2|$, and that

$$y_1 \sim 1 \text{ and } |y_2 - cy_1^{a_\lambda}| \lesssim \varepsilon_0$$

for y in the support of the integrand of $J_k(\xi)$. Recall also that we are assuming that $|\xi| \sim |\xi_3|$.

One is thus led to the *estimation of oscillatory integrals depending on certain parameters* (here s_1, s_2, S_2) which may have various relative sizes.

The case where $|S_2| \geq M$ for some sufficiently large constant $M \gg 1$ can easily be treated by means of van der Corput's lemma applied to the y_1 -integration, so let us assume that $|S_2| < M$. Then $|s_2| \ll 1$, provided we have chosen k_0 sufficiently large.

We may also easily reduce to the case where

$$(3.15) \quad \partial_2^j \phi_{\text{pr}}^a(1, c) = \partial_2^j \phi_{\kappa^\lambda}^a(1, c) = 0 \text{ for } 1 \leq j < h,$$

for otherwise an integration by parts in y_2 (if $j = 1$) or a simple application of Lemma 3.2 yields a suitable estimate as before.

The case where $c > 0$ can easily be reduced to the case $c = 0$ by performing another change of variables $y_2 \mapsto y_2 + cy_1^{a_\lambda}$ in the integral defining $J_k(\xi)$. Indeed, one can show that our assumption (3.15) implies that necessarily $a_\lambda = \kappa_2^\lambda / \kappa_1^\lambda \in \mathbb{N}$ (cf. Corollary 3.2 (iii) in [38]), and one checks that the new coordinates are again adapted to ϕ .

So, let us assume that $c = 0$. Then necessarily $\phi_{\text{pr}}^a(1, 0) \neq 0$, for otherwise ϕ_{pr}^a would have a root of multiplicity at least h at $(1, 0)$, which would contradict (3.12).

Assuming without loss of generality that $\phi_{\text{pr}}^a(1, 0) = 1$, we can then write (compare [38], Subsection 9.1)

$$\phi_{\text{pr}}^a(y_1, y_2) = y_2^B Q(y_1, y_2) + y_1^n,$$

where Q is a κ^λ -homogeneous polynomial such that $Q(1, 0) \neq 0$, and where $B \geq h > 2$.

Recall that $|S_2| < M$, so that $|s_2| \ll 1$. Moreover, the case where $|s_1| \geq N$ for some sufficiently large constant $N \gg 1$ can easily be dealt with by means of an integration by parts in y_1 , so let us assume that $|s_1| < N$.

It will then suffice to show that, given any point $(s_1^0, S_2^0) \in [-M, M] \times [-N, N]$ and any point $y_1^0 \sim 1$, there exist a neighborhood U of (s_1^0, S_2^0) , a neighborhood V of y_1^0 and some $\sigma > 1/h$ such that we have an estimate of the form

$$(3.16) \quad |J_k(\xi)| \lesssim \frac{2^{-k|\kappa|}}{(1 + 2^{-k}|\xi|)^\sigma}$$

for every $(s_1, S_2) \in U$, provided the function χ in the definition of $J_k(\xi)$ is supported in V and ε_0 and k are chosen sufficiently small, respectively large. Summing over all k , this will clearly imply an estimate as in (3.3), even without logarithmic factor.

To this end, first notice that for $(s_1, S_2) \in U$ and k sufficiently large, the function $F_k(y, s)$ can be viewed as a small C^∞ -perturbation of the function

$$F_{\text{pr}}(y) := y_2^B Q(y_1, y_2) + s_1^0 y_1 + S_2^0 \omega(0) y_1^m + y_1^n.$$

Thus, if $\nabla F_{\text{pr}}(y_1^0, 0) \neq 0$, then we obtain (3.16), with $\sigma = 1$, simply by integration by parts.

Assume therefore that $(y_1^0, 0)$ is a critical point of F_{pr} . Then y_1^0 is a critical point of the polynomial function

$$g(y_1) := s_1^0 y_1 + S_2^0 \omega(0) y_1^m + y_1^n,$$

which comprises all terms of F_{pr} depending on the variable y_1 only. Note that the exponents satisfy $1 < m < n$, since $n = 1/\kappa_1^\lambda > \kappa_2^\lambda/\kappa_1^\lambda > m$. It is then easy to see that g'' and g''' cannot vanish simultaneously at the given point y_1^0 , so that there are positive constants $c_1, c_2 > 0$ and a compact neighborhood V of $(y_1^0, 0)$ such that

$$c_1 \leq \sum_{j=2}^3 |g^{(j)}(y_1)| \leq c_2 \quad \text{for every } y_1 \in V.$$

This implies an analogous estimate for the partial derivatives $\partial_1^j F_k(y_1, y_2, s)$ of F_k , uniformly for $(s_1, S_2) \in U$ and the range of y_2 that we consider, provided we choose U and ε_0 sufficiently small. Applying the van der Corput type estimate in Lemma 3.2, we thus obtain estimate (3.16), with $\sigma = 1/3$, so that we are done provided $h > 3$. Notice also that if $g''(y_1^0) \neq 0$, then by the same type of argument we see that (3.16) holds true with $\sigma = 1/2 > 1/h$.

We may thus finally assume that $2 < h \leq 3$, and that $g'(y_1^0) = g''(y_1^0) = 0$. In this case we have

$$\frac{1}{\kappa_1^\lambda + \kappa_2^\lambda} = h \leq 3 \quad \text{and} \quad \frac{\kappa_2^\lambda}{\kappa_1^\lambda} > m \geq 2,$$

so that $1/\kappa_2^\lambda < 9/2$.

Note that $B \leq 1/\kappa_2^\lambda$ is a positive integer, and $h \leq B < 9/2$, so that either $B = 4$ or $B = 3$. We translate the critical point $(y_1^0, 0)$ of F_{pr} to the origin by considering the function

$$F_{\text{pr}}^\#(z) := F_{\text{pr}}(y_1^0 + z_1, z_2) - g(y_1^0) = z_2^B Q(y_1^0 + z_1, z_2) + \frac{1}{6} g^{(3)}(y_1^0) z_1^3 + \dots$$

It is easy to see that this function has height $h^\# := h(F_{\text{pr}}^\#)$ given by $h^\# = \frac{1}{1/3+1/B} < 2$. We can therefore again apply Duistermaat's results in [24] to the corresponding part of the oscillatory integral $J_k(\xi)$ and obtain estimate (3.16), with $\sigma = 1/h^\# > 1/h$. Note here that the estimates in [24] are stable under small perturbations.

4. PROBLEM B: L^p -BOUNDEDNESS OF THE MAXIMAL OPERATOR \mathcal{M} ASSOCIATED TO THE HYPERSURFACE S .

We next turn to Problem B. As has already been mentioned in the introduction, the first, fundamental result on this problem is due to Stein [61], who proved that for $n \geq 3$, the spherical maximal function is bounded on $L^p(\mathbb{R}^n)$ for every $p > n/(n-1)$. The analogous estimate in dimension $n = 2$ turned to require even deeper methods and was later established by Bourgain [9]. These results became the starting point for intensive studies of various classes of maximal operators associated to subvarieties. Stein's monograph [62] is an excellent reference to many of these developments. From these early works, the influence of geometric properties of S on the validity of L^p -estimates for the maximal operator \mathcal{M} became evident. For instance, Greenleaf [34] proved that \mathcal{M} is bounded on $L^p(\mathbb{R}^n)$ if $n \geq 3$ and $p > n/(n-1)$, provided S has everywhere non-vanishing Gaussian curvature. In contrast, the case where the Gaussian

curvature vanishes at some point is still wide open, and complete answers are available at present only for finite type curves in the plane [43], and finite type hypersurfaces in \mathbb{R}^3 and $p > 2$ (cf. Theorem 4.1).

Let me come back to our hypersurface S in \mathbb{R}^3 . Recall from the introduction that we assume that S satisfies the transversality condition of §1, so that, by localizing to a small neighborhood of a given point $x^0 \in S$ and applying a suitable linear change of coordinates, we may assume that $x^0 = 0$ and that S is given as the graph $S = \text{graph}(1 + \phi)$, where ϕ is a smooth, finite type function defined on a neighborhood of the origin and satisfying $\phi(0, 0) = 0, \nabla\phi(0, 0) = 0$. We then define the height of S at x^0 by

$$h(x^0, S) := h(\phi).$$

It easily seen that this notion is invariant under affine linear changes of coordinates in the ambient space \mathbb{R}^3 . Recall also that $\mathcal{M}f(x) := \sup_{t>0} |A_t f(x)|$, where A_t denotes the averaging operator over the t -dilate of S given by

$$A_t f(x) := \int_S f(x - ty) \rho(y) d\sigma(y), \quad t > 0.$$

We can then state our main result from [38] (Theorems 1.2, 1.3), which gives an almost complete answer to the question of L^p -boundedness of the maximal operator \mathcal{M} when $p > 2$:

Theorem 4.1. *Assume the hypersurface S satisfies the transversality Assumption 1.1.*

- (i) *If the measure $\rho d\sigma$ is supported in a sufficiently small neighborhood of x^0 , then \mathcal{M} is bounded on $L^p(\mathbb{R}^3)$ whenever $p > \max\{h(x^0, S), 2\}$.*
- (ii) *If \mathcal{M} is bounded on $L^p(\mathbb{R}^3)$ for some $p > 1$, and if $\rho(x^0) > 0$, then $p \geq h(x^0, S)$. Moreover, if S is analytic at x^0 , then $p > h(x^0, S)$.*

4.1. Related quantities: contact index and sublevel growth. In [44], Iosevich and Sawyer had discovered a very interesting connection between the behavior of maximal functions \mathcal{M} associated to hypersurfaces S (in \mathbb{R}^n) and an integrability index associated to the hypersurface. In order to describe this, assume that $d_H(x) := \text{dist}(H, x)$ denotes the distance from a point x on S to a given hyperplane H . In particular, if $x^0 \in S$, then $d_{T, x^0}(x) := \text{dist}(x^0 + T_{x^0}S, x)$ will denote the distance from $x \in S$ to the affine tangent plane to S at the point x^0 . The following result has been proved in [44] in arbitrary dimensions $n \geq 2$ and without requiring Assumption 1.1.

Theorem 4.2 (Iosevich-Sawyer). *If the maximal operator \mathcal{M} is bounded on $L^p(\mathbb{R}^n)$, where $p > 1$, then*

$$(4.1) \quad \int_S d_H(x)^{-1/p} \rho(x) d\sigma(x) < \infty$$

for every affine hyperplane H in \mathbb{R}^n which does not pass through the origin.

Moreover, it was conjectured in [44] that for $p > 2$ the condition (4.1) is indeed necessary and sufficient for the boundedness of the maximal operator \mathcal{M} on L^p , at least if for instance S is compact and $\rho > 0$.

Remark 4.3. *Notice that condition (4.1) is easily seen to be true for every affine hyperplane H which is nowhere tangential to S , so that it is in fact a condition on affine tangent hyperplanes to S only. Moreover, if Assumption 1.1 is satisfied, then there are no affine tangent hyperplanes which pass through the origin, so that in this case it is a condition on all affine tangent hyperplanes.*

Moreover, it is not very hard to prove (cf. [38]) that if S is a smooth hypersurface of finite type in \mathbb{R}^3 , then, for every $p < h(x^0, S)$,

$$(4.2) \quad \int_{S \cap U} d_{T, x^0}(x)^{-1/p} d\sigma(x) = \infty \quad \text{for every } p < h(x^0, S)$$

and every neighborhood U of x^0 . And, if S is analytic near x^0 , then (4.2) holds true also for $p = h(x^0, S)$.

Notice that this result does not require Assumption 1.1. As an immediate consequence of Theorem 4.1, Theorem 4.2 and (4.2) we obtain

Corollary 4.4. *Assume that $S \subset \mathbb{R}^3$ is of finite type and satisfies Assumption 1.1, and let $x^0 \in S$ be a fixed point. Moreover, let $p > 2$.*

Then, if S is analytic near x^0 , there exists a neighborhood $U \subset S$ of the point x^0 such that for any $\rho \in C_0^\infty(U)$ with $\rho(x^0) > 0$ the associated maximal operator \mathcal{M} is bounded on $L^p(\mathbb{R}^3)$ if and only if condition (4.1) holds for every affine hyperplane H in \mathbb{R}^3 which does not pass through the origin.

If S is only assumed to be smooth near x^0 , then the same conclusion holds true, with the possible exception of the exponent $p = h(x^0, S)$.

This confirms the conjecture by Iosevich and Sawyer in our setting for analytic S , and for smooth, finite-type S with the possible exception of the exponent $p = h(x^0, S)$. For the critical exponent $p = h(x^0, S)$, if S is not analytic near x^0 , examples show that unlike in the analytic case it may happen that \mathcal{M} is bounded on $L^{h(x^0, S)}$ (see, e.g., [45]), and the conjecture remains open for this value of p .

In view of these results, it is natural to define the *uniform contact index* $\gamma_u(x^0, S)$ of the hypersurface S at the point $x^0 \in S$ as the supremum over the set of all γ for which there exists an open neighborhood U of x^0 in S such that the estimate

$$\int_{U \cap S} d_H(x)^{-\gamma} d\sigma(x) < \infty$$

holds true for every affine hyperplane H in \mathbb{R}^n . If we restrict ourselves in this definition to the affine tangent hyperplane $H = x^0 + T_{x^0}S$ at the point x^0 , we shall call the corresponding index the *contact index* $\gamma(x^0, S)$ of the hypersurface S at the point $x^0 \in S$. Here, we shall always assume that $\rho(x^0) \neq 0$. Note that if we change coordinates so that $x^0 = 0$ and S is the graph of ϕ near the origin, where ϕ satisfies (1.2), then the

contact index is just the supremum over all γ such that there is some neighborhood U of the origin so that $|\phi|^{-\gamma} \in L^1(U)$, so that the contact index agrees with the *critical integrability index* of ϕ as defined for instance in [15].

A closely related quantity is the *sublevel growth rate* $\sigma(\phi)$, defined as the supremum over all $\sigma > 0$ such that there is some constant $C_\sigma > 0$ so that

$$\left| \{x \in U : |\phi(x)| < \varepsilon\} \right| \leq C_\sigma \varepsilon^\sigma \quad \text{for every } \varepsilon > 0.$$

Indeed, it can easily be shown by means of Tchebychev's inequality (cf. [15]) that

$$\sigma(\phi) = \gamma(x^0, S), \quad \text{if } x^0 = 0 \text{ and } S = \text{graph}(\phi).$$

In analogy with Arnol'd's notion of "singularity index" [2], let us finally introduce the *uniform oscillation index* $\beta_u(x^0, S)$ of the hypersurface S at the point $x^0 \in S$ as the supremum over the set of all β such that

$$|\widehat{\rho d\sigma}(\xi)| \leq C_\beta (1 + |\xi|)^{-\beta},$$

for all ρ supported in a sufficiently small neighborhood of x^0 . If we restrict directions ξ to the normal to S at x^0 , then the corresponding decay rate will be called the *oscillation index* $\beta(x^0, S)$ at x^0 .

Combining our results with results from [56] (compare also [27]), we easily obtain the following result for smooth, finite type hypersurfaces S in \mathbb{R}^3 :

Corollary 4.5. *Let $x^0 \in S \subset \mathbb{R}^3$ be a given point so that $\rho(x^0) > 0$. Then*

$$\beta_u(x^0, S) = \beta(x^0, S) = \gamma_u(x^0, S) = \gamma(x^0, S) = 1/h(x^0, S).$$

Remarks 4.6. (a) *In dimension $n \geq 3$, the corresponding identities may fail to be true. Indeed, Varchenko had already observed in [66] (Example 3) that for graphs S of functions ϕ of three variables the oscillation index $\beta(0, S)$ may differ from $1/h(0, S)$. Moreover, it is easy to give examples of functions ϕ of three variables where the contact index $\gamma = \gamma(0, S)$ is strictly smaller than the oscillation index $\beta = \beta(0, S)$. For instance, this applies to $\phi(x_1, x_2, x_3) := x_3^2 - (x_1^2 + x_2^2)$, where the method of stationary phase shows that $\beta = 3/2$, whereas the factorization $\phi(x_1, x_2, x_3) = (x_3 - \sqrt{x_1^2 + x_2^2})(x_3 + \sqrt{x_1^2 + x_2^2})$ shows that $\gamma = 1$ (compare counterexample 8.1 in [15]).*

However, the proof of Theorem 1.6 in [31] shows that the oscillation index β can only be different from the contact index γ when $1/\gamma$ is an odd integer.

To the best of my knowledge, it is not known whether the identities $\beta_u(x^0, S) = \beta(x^0, S)$ and $\gamma_u(x^0, S) = \gamma(x^0, S)$ may persist in higher dimensions. Indeed, this question represents a special case of a question posed by Arnol'd, namely whether the oscillation index of a given function ϕ is semicontinuous in the following sense:

Suppose $\phi(x, s)$ is a phase function which depends on x in a small neighborhood of the origin in \mathbb{R}^n and some small parameters $s \in \mathbb{R}^k$. Is there is a small neighborhood of $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^k$ so that the oscillation index of $\phi(x, s)$

at x^0 is greater or equal to that of $\phi(x, 0)$ at the origin for every (x^0, s^0) in this neighborhood? That question had been answered to the negative in dimensions $n \geq 3$ by Varchenko [66], and to the positive for analytic functions depending on two variables by Karpushkin [47].

For linear perturbations of a given function ϕ , our Theorem 3.1 shows that Arnol'd's conjecture holds true even for smooth, finite type functions ϕ , and it seems open whether this kind of stability may still hold true even in higher dimensions, since all counter examples to Arnol'd's conjecture which have been found hitherto have non-linear perturbation terms.

Further important papers dealing with this stability question for general perturbations are, e.g., [13], [56].

- (b) Greenblatt [28],[30], and independently Collins, Greenleaf and Paramanik [15], have devised (quite different) algorithms of resolution of singularities which in principle allow to compute the contact index also in higher dimensions.
- (c) If $p \leq 2$, then examples (see, e.g., [46]) show that neither the notion of height nor that of contact index will determine the range of exponents p for which the maximal operator \mathcal{M} is L^p -bounded. Our work (in progress) on this case seems to indicate a certain conjecture how to express this range in terms of Newton polyhedra. Moreover, for certain surfaces this conjecture relates to fundamental open problems in Fourier analysis, such as the conjectured reverse square function estimate for the cone multiplier (see, e.g., [49], [10]).

4.2. A few hints on the proof of Theorem 4.1. Since the proof in [38] is quite involved, I shall here just try to indicate some of the main steps, grossly oversimplifying some of the arguments.

In the first part of the proof, we basically follow the first two steps of the scheme that has been outlined in the preceding paragraph. Recall that in these steps, it had effectively been possible to reduce the problem to a one-dimensional one. The same applies basically to the estimation of the maximal operator \mathcal{M} , because in the first two steps, we have had a sufficiently good control on the multiplicity of roots by the height h .

In these arguments, the following result plays an analogous role for problem B as the van der Corput type Lemma 3.2 played for problem A.

Let U be an open neighborhood of the point $x^0 \in \mathbb{R}^2$, and let $\phi_{\text{pr}} \in C^\infty(U, \mathbb{R})$ such that

$$(4.3) \quad \partial_2^m \phi_{\text{pr}}(x_1^0, x_2^0) \neq 0,$$

where $m \geq 2$. Let

$$\phi = \phi_{\text{pr}} + \phi_r,$$

where $\phi_r \in C^\infty(U, \mathbb{R})$ is a sufficiently small perturbation. Denote by S_ε the surface in \mathbb{R}^3 given by $S_\varepsilon := \{(x_1, x_2, 1 + \varepsilon\phi(x_1, x_2)) : (x_1, x_2) \in U\}$, with $\varepsilon > 0$, and consider the

averaging operators

$$A_t^\varepsilon f(x) := \int_{S_\varepsilon} f(x - ty) \psi(y) d\sigma(y),$$

where $d\sigma$ denotes the surface measure and $\psi \in C_0^\infty(S_\varepsilon)$ is a non-negative cut-off function. Define the associated maximal operator by

$$\mathcal{M}^\varepsilon f(x) := \sup_{t>0} |A_t^\varepsilon f(x)|.$$

Proposition 4.7. *Assume that ϕ_{pr} satisfies (4.3) and that the neighborhood U of the point x^0 is sufficiently small. Then there exist numbers $M \in \mathbb{N}$, $\delta > 0$, such that for every $\phi_r \in C^\infty(U, \mathbb{R})$ with $\|\phi_r\|_{C^M} < \delta$ and every $p > m$ there exists a positive constant C_p such that for $\varepsilon > 0$ sufficiently small the maximal operator \mathcal{M}^ε satisfies the following a priori estimate:*

$$(4.4) \quad \|\mathcal{M}^\varepsilon f\|_p \leq C_p \varepsilon^{-1/p} \|f\|_p, \quad f \in \mathcal{S}(\mathbb{R}^3).$$

This result can be reduced to the one-dimensional case, i.e., the study of maximal functions associated to curves in the plane. Indeed, just consider the "fan" of all hyperplanes passing through the x_1 -axis. This fan will fibre the given surface into a family of curves, and our dilations leave each of these planes invariant.

The proof for the analogous result for plane curves then basically follows Iosevich's approach in [43]. The case where $m = 2$ is indeed the most difficult one; the cases where $m \geq 3$ can then easily be reduced to this case by means of dyadic decompositions and re-scalings. Note that if $m = 2$, then the corresponding maximal operator in the plane behaves in a similar way as the "circular" maximal function studied by Bourgain [9]. An alternative approach to Bourgain's one had later been given by Mockenhaupt, Seeger and Sogge [51], [50], based on suitable local smoothing estimates for classes of Fourier integral operators. This approach is stable under small perturbations of the given curve, which is exactly what is needed for our purposes.

Indeed, in our applications of Proposition 4.7, we even need to replace $\phi(x_1, x_2)$ by $\phi(x_1, x_2 - \psi_\varepsilon(x_1))$, where the function ψ_ε may blow up like $O(\varepsilon^{-\delta})$ for some $\delta \in [0, 1[$, and it turns out that this is still admissible, i.e., the estimate (4.4) remains valid also in this case.

Now, if the coordinates are *adapted* to ϕ (a similar argument will apply to the first step when the coordinates are not adapted), then, in analogy with the decomposition $J(\xi) = \sum_{k=k_0}^\infty J_k(\xi)$, we can dyadically decompose the surface S as in Subsection 3.1.1, and accordingly decompose

$$A_t f(y, y_3) = \sum_{k=k_0}^\infty A_t^k f(y, y_3),$$

where one finds that

$$A_t^k f(y, y_3) = 2^{-k|\kappa|} \int_{\mathbb{R}^2} f\left(y - t\delta_{2^{-k}}(x), y_3 - t(1 + 2^{-k}\phi^k(x))\right) \eta(\delta_{2^{-k}}x) \chi(x) dx,$$

with ϕ^k is as before. We denote by \mathcal{M}^k the maximal operator associated to the averaging operators A_t^k . Observe next that the scaling operators T^k , defined by

$$T^k f(y, y_3) := 2^{\frac{k|\kappa|}{p}} f(\delta_{2^k}(y), y_3),$$

act isometrically on $L^p(\mathbb{R}^3)$, and

$$(T^{-k} A_t^k T^k) f(y, y_3) = 2^{-k|\kappa|} \int_{\mathbb{R}^2} f\left(y - tx, y_3 - t(1 + 2^{-k}\phi^k(x))\right) \eta(\delta_{2^{-k}}(x)) \chi(x) dx,$$

so that we are reduced to estimating the maximal operator associated to this family of averaging operators. This, in return, can be accomplished by means of Proposition 4.7, where we choose $\varepsilon := 2^{-k}$. The resulting estimates can then be summed over k , and we arrive at the desired estimation for \mathcal{M} in this case.

Assume next that the coordinates x are *not adapted* to ϕ . Eventually, we then again arrive at the situation studied in the third step (compare Subsection 3.1.3), i.e., we need to estimate the contribution of the domain D_λ to the maximal operator \mathcal{M} . Again, one can indeed reduce to a small subdomain of the form

$$|x_2 - \psi(x_1) - cx_1^{a_\lambda}| \leq \varepsilon_0 x_1^{a_\lambda}, \quad \text{with } x_1 > 0,$$

and also assume that (3.15) holds true at the point $(1, c) = (1, 0)$. Recall that we may then write

$$\phi_{\text{pr}}^a(y_1, y_2) = y_2^B Q(y_1, y_2) + y_1^n,$$

where Q is a κ^λ -homogeneous polynomial such that $Q(1, 0) \neq 0$, and $B \geq h > 2$. Again, in this situation, a reduction to the one-dimensional case is no longer possible.

It turns out that this is the most difficult case, which requires various further ideas. The heart of the matter are in fact rather precise estimations of certain classes of two-dimensional oscillatory integrals depending on small parameters (cf. §5 in [38]).

Indeed, in order to understand the behavior of ϕ^a as a function of y_2 , for y_1 fixed, we decompose

$$\phi^a(y_1, y_2) = \phi^a(y_1, 0) + \theta(y_1, y_2),$$

and write the complete phase Φ for $J(\xi)$ in adapted coordinates (y_1, y_2) in the form

$$F(y, \xi) = (\xi_3 \phi^a(y_1, 0) + \xi_1 y_1 + \xi_2 \psi(y_1)) + (\xi_3 \theta(y_1, y_2) + \xi_2 y_2).$$

Notice that

$$\phi^a(y_1, 0) = y_1^n \rho(y_1), \quad \psi(y_1) = y_1^{m_1} \omega(y_1), \quad \theta_{\kappa^\lambda}(y_1, y_2) = y_2^B Q(y_1, y_2),$$

where θ_{κ^λ} denotes the κ^λ -principal part of θ , and where ρ and ω do not vanish at $y_1 = 0$.

Now, by means of a dyadic decomposition and re-scaling using the κ^λ -dilations $\{\delta_r^\lambda\}_{r>0}$ we would like to reduce our considerations as before to the domain where $y_1 \sim 1$. In this domain, $|y_2| \ll 1$, so that $\theta_{\kappa^\lambda}(y) \sim y_2^B Q(y_1, 0)$. What leads to problems is that the “error term” $\theta_r := \theta - \theta_{\kappa^\lambda}$, which consists of terms of higher κ^λ -degree than

$\theta_{\kappa}^{\lambda}$, may nevertheless contain terms of lower y_2 -degree $l_j < B$ of the form $c_j y_2^{l_j} y_1^{n_j}$, provided n_j is sufficiently large (this corresponds to a “fine splitting of roots” of θ when ϕ is analytic). After scaling the k -th dyadic piece in our decomposition by $\delta_{2^{-k}}^{\lambda}$ in order to achieve that $y_1 \sim 1$ and $|y_2| \lesssim \varepsilon_0$, such terms will have small coefficients compared to the one of $y_2^B Q(y_1, y_2)$, but for $|y_2|$ very small they may nevertheless become dominant and have to be taken into account.

In order to resolve this problem, we apply a *further domain decomposition* by means of a suitable stopping time argument, into homogeneous domains D'_ℓ and transition domains E'_ℓ , oriented, in some sense, at the level sets of $\partial_2 \phi^a$, which in return again are chopped up into dyadic respectively bi-dyadic pieces. After re-scaling, the contributions of these pieces to the maximal operators can eventually be estimated by means of *oscillatory integral technics in two variables*.

More precisely, it turns out that what is needed are *uniform estimates* for various classes of oscillatory integrals of the form

$$J(\lambda, \sigma, \delta) := \int_{\mathbb{R}^2} e^{i\lambda F(x, \sigma, \delta)} a(x, \delta) dx, \quad (\lambda > 0),$$

with a phase function F of the form

$$F(x_1, x_2, \sigma, \delta) := f_1(x_1, \delta) + \sigma f_2(x_1, x_2, \delta),$$

and an amplitude a defined for x in some open neighborhood of the origin in \mathbb{R}^2 with compact support in x . The functions f_1, f_2 are assumed to be real-valued and depend, like the function a , smoothly on x and on small real parameters $\delta_1, \dots, \delta_\nu$, which form the vector $\delta := (\delta_1, \dots, \delta_\nu) \in \mathbb{R}^\nu$. σ denotes another small real parameter.

With a slight abuse of language we shall say that ψ is compactly supported in some open set $U \subset \mathbb{R}^2$ if there is a compact subset $K \subset U$ such that $\text{supp } \psi(\cdot, \delta) \subset K$ for every δ .

To give an idea as to which type of oscillatory integrals we need to estimate, let us remark that the most difficult instance are “oscillatory integrals of degenerate Airy type”, for which we have the following result:

Theorem 4.8. *Assume that*

$$|\partial_1 f_1(0, 0)| + |\partial_1^2 f_1(0, 0)| + |\partial_1^3 f_1(0, 0)| \neq 0 \quad \text{and} \quad \partial_1 \partial_2 f_2(0, 0, 0) \neq 0,$$

and that there is some $m \geq 2$ such that

$$\partial_2^l f_2(0, 0, 0) = 0 \text{ for } l = 1, \dots, m-1 \text{ and } \partial_2^m f_2(0, 0, 0) \neq 0.$$

Then there exist a neighborhood $U \subset \mathbb{R}^2$ of the origin and constants $\varepsilon, \varepsilon' > 0$ such that for any amplitude a which is compactly supported in U the following estimate

$$(4.5) \quad |J(\lambda, \sigma, \delta)| \leq \frac{C}{\lambda^{\frac{1}{2}+\varepsilon} |\sigma|^{(l_m + c_m \varepsilon)}}$$

holds true uniformly for $|\sigma| + |\delta| < \varepsilon'$, where $l_m := \frac{1}{6}$ and $c_m := 1$ for $m < 6$, and $l_m := \frac{m-3}{2(2m-3)}$ and $c_m := 2$ for $m \geq 6$.

Observe that the order of decay $O(\lambda^{-1/2-\varepsilon})$ in (4.5) is just what we need in order to apply the usual method to control the maximal operator on L^2 by means of Sobolev's embedding theorem applied to the scaling variable $t > 0$.

Remark 4.9. *If the Fourier transform of the surface carried measure μ decays more slowly than $O(|\xi|^{-1/2-\varepsilon})$, then one cannot directly apply Sobolev's embedding theorem in order to control the maximal function on L^2 . This happens in many situations when the Gaussian curvature of the hypersurface S vanishes. One method to overcome this problem is to use suitable damping factors in the amplitude of the corresponding oscillatory integrals, for instance powers of the Gaussian curvature, and combine this with a suitable complex interpolation argument. This technique had been introduced in [60], and also been used in our first approach our problems in [37].*

However, the choice of a suitable damping factor can become quite tricky a task, and we believe that the techniques in [38], which avoid damping factors and rather rely on suitable decompositions of the given surface in combination with re-scaling arguments, are simpler and more straight-forward.

For an approach based on damping methods, we refer to [32], where most of those cases are treated by damping techniques which can be reduced to a one-dimensional problem.

5. PROBLEM C: FOURIER RESTRICTION TO THE HYPERSURFACE S .

We finally turn to the last problem, namely the determination of the range of exponents p for which an L^p - L^2 Fourier restriction estimate

$$(5.1) \quad \left(\int_S |\hat{f}(x)|^2 d\mu(x) \right)^{1/2} \leq C \|f\|_{L^p(\mathbb{R}^3)}, \quad f \in \mathcal{S}(\mathbb{R}^3),$$

holds true.

The idea of Fourier restriction goes back to Stein, and a first instance of this concept is the sharp L^p - L^q Fourier restriction estimate for the circle in the plane by Zygmund [67], who extended earlier work by Fefferman and Stein [25] (see also Hörmander [36], [14] for estimates on more general oscillatory integral operators). For subvarieties of higher dimension, the first fundamental result was obtained (in various steps) for Euclidean spheres S^{n-1} by Stein and Tomas, who eventually proved that an L^p - L^2 Fourier restriction estimate holds true for S^{n-1} , $n \geq 3$, if and only if $p' \geq 2(2/(n-1)+1)$ (cf. [62] for the history of this result).

Even though I shall not pursue that problem here, let me briefly remind that a more general and even substantially deeper problem is to determine the exact range of exponents p and q for which an L^p - L^q restriction estimate

$$\left(\int_{S^{n-1}} |\hat{f}(x)|^q d\sigma(x) \right)^{1/q} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

holds true for spheres. It is conjectured that this is the case if and only if $p' > 2n/(n-1)$ and $p' \geq q(2/(n-1)+1)$, and there has been a lot of very deep work on this problem

by many mathematicians, including Bourgain, Wolff, Vargas, Vega, Katz, Tao, Keel, Lee, and most recently Bourgain and Guth [11], which has led to important progress, but the problem is still open in dimensions $n \geq 3$ and represents one of the major challenges in Euclidean harmonic analysis, bearing various deep connections with other important open problems, such as the Bochner-Riesz conjecture, the Kakeya conjecture and Sogge's local smoothing conjecture for solutions to the wave equation. I refer to Stein's book [62] for more information on these topics and their history until 1993, and various related essays by T. Tao which can be found on his webpage.

Coming back to the restriction estimate (5.1) for our hypersurface S in \mathbb{R}^3 , we begin with the case where there exists a linear coordinate system which is adapted to the function ϕ . For this case, a complete answer had been given in [40] (for analytic hypersurfaces, partial results had been obtained before by Magyar [48]) :

Theorem 5.1. *Assume that, after applying a suitable linear change of coordinates, the coordinates (x_1, x_2) are adapted to ϕ . We then define the critical exponent p_c by*

$$(5.2) \quad p'_c := 2h(\phi) + 2,$$

where p' denotes the exponent conjugate to p , i.e., $1/p + 1/p' = 1$.

Then there exists a neighborhood $U \subset S$ of the point x^0 such that for every non-negative density $\rho \in C_0^\infty(U)$ the Fourier restriction estimate (5.1) holds true for every p such that

$$(5.3) \quad 1 \leq p \leq p_c.$$

Moreover, if $\rho(x^0) \neq 0$, then the condition (5.3) on p is also necessary for the validity of (5.1).

In many cases, this result is an immediate consequence of Theorem 3.1 and Greenleaf's classical restriction estimate in [34], which I shall state here for the special case of hypersurfaces only.

Theorem 5.2 (Greenleaf). *Assume that $|\widehat{d\mu}(\xi)| \lesssim |\xi|^{-1/h}$. Then the restriction estimate (5.1) holds true for every $p \geq 1$ such that $p' \geq 2(h+1)$.*

Indeed, observe that a problem arises only when Varchenko's exponent $\nu(\phi)$ equals one in (3.3). In this case, a direct application of Greenleaf's result yields only the range $1 \leq p < p_c$.

To capture also the endpoint $p = p_c$, recall from Subsection 3.1.1 that we had effectively decomposed the measure μ into a dyadic sum $\mu = \sum_{k=k_0}^\infty \mu_k$, where $\mu_k = (\chi_k \otimes 1)\mu$, and where $\hat{\mu}_k(\xi) = J_k(\xi)$ is given by (3.4). Moreover, we had estimated $J_k(\xi)$ in (3.5), which in particular yields the estimate

$$|\widehat{\mu}_k(\xi)| \leq C 2^{-k|\kappa|} (1 + 2^{-k} |\xi_3|)^{-1/h}.$$

The measures μ_k are supported in dyadic annuli of "radius" 2^{-k} , which are images of an annulus of radius of size one under the dilations $\delta_{2^{-k}}$ associated to the principal weight κ , so that we cannot directly apply Greenleaf's restriction estimate. However, it

is important to notice that our estimates for $\widehat{\mu_k}(\xi)$ do not carry a logarithmic factor yet (that only arose for $\hat{\mu}(\xi)$ in certain cases through summation over the k), and a simple re-scaling argument can then be applied to derive the following uniform restriction estimate for the family of measures μ_k :

$$(5.4) \quad \int |\hat{f}(x)|^2 d\mu_k(x) \leq C^2 \|f\|_{p_c}^2.$$

Fix a cut-off function $\tilde{\chi} \in C_0^\infty(\mathbb{R}^2)$ supported in an annulus centered at the origin such that $\tilde{\chi} = 1$ on the support of χ , and define dyadic frequency decomposition operators Δ'_k by

$$\widehat{\Delta'_k f}(x) := \tilde{\chi}(\delta_{2^k} x') \hat{f}(x', x_3),$$

where we have written $(x_1, x_2) = x'$. Then $\int |\hat{f}(x)|^2 d\mu_k(x) = \int |\widehat{\Delta'_k f}(x)|^2 d\mu_k(x)$, and setting $p := p_c$, we see that (5.4) yields in fact

$$\int |\hat{f}(x)|^2 d\mu_k(x) \leq C^2 \|\widehat{\Delta'_k f}\|_p^2,$$

for any $k \geq k_0$. In combination with Minkowski's inequality, this implies

$$\begin{aligned} \left(\int |\hat{f}(x)|^2 d\mu(x) \right)^{1/2} &= \left(\sum_{k \geq k_0} \int |\hat{f}(x)|^2 d\mu_k(x) \right)^{1/2} \leq \left(\sum_{k \geq k_0} \|\Delta'_k f\|_p^2 \right)^{1/2} \\ &= C \left(\left(\sum_{k \geq k_0} \left(\int |\Delta'_k f(x)|^p dx \right)^{2/p} \right)^{p/2} \right)^{1/p} \leq C \left\| \left(\sum_{k \geq k_0} |\Delta'_k f(x)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)}, \end{aligned}$$

since $p < 2$. Estimate (5.1) for $p = p_c$ follows thus by means of Littlewood-Paley theory.

From now on, we shall therefore always make the following

Assumption 5.3. *There is no linear coordinate system which is adapted to ϕ .*

According to our discussion in Subsection 2.1, we may then also assume that the coordinates x are linearly adapted to ϕ , and that there are adapted coordinates y of the form $y_1 = x_1, y_2 = x_2 - \psi(x_1)$, where

$$(5.5) \quad \psi(x_1) = x_1^m \omega(x_1), \quad \text{with } \omega(0) \neq 0 \text{ and } m \geq 2.$$

$\phi^a(y)$ will again denote ϕ when expressed in these adapted coordinates, and we shall use the notions introduced for the study of the Newton polyhedron of ϕ^a in Subsection 3.1.2.

Consider the line parallel to the bi-sectrix

$$\Delta^{(m)} := \{(t, t + m + 1) : t \in \mathbb{R}\}.$$

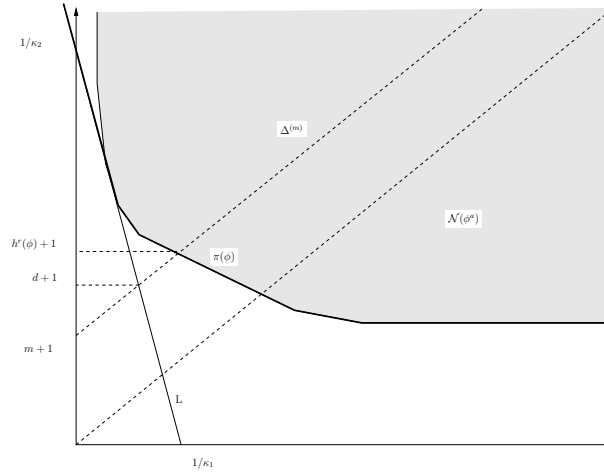


FIGURE 4. r-height

For any edge $\gamma_l \subset L_l := \{(t_1, t_2) \in \mathbb{R}^2 : \kappa_1^l t_1 + \kappa_2^l t_2 = 1\}$ define h_l by

$$\Delta^{(m)} \cap L_l = \{(h_l - m, h_l + 1)\},$$

i.e.,

$$(5.6) \quad h_l = \frac{1 + m\kappa_1^l - \kappa_2^l}{\kappa_1^l + \kappa_2^l},$$

and define the *restriction height*, or short, *r-height*, of ϕ by

$$h^r(\phi) := \max(d, \max_{\{l=1, \dots, n+1 : a_l > m\}} h_l).$$

Remarks 5.4. (a) For L in place of L_l and κ in place of κ^l , one has $m = \kappa_2/\kappa_1$ and $d = 1/(\kappa_1 + \kappa_2)$, so that one gets d in place of h_l in (5.6).

(b) Since $m < a_l$, we have $h_l < 1/(\kappa_1^l + \kappa_2^l)$, hence $h^r(\phi) < h(\phi)$.

It is easy to see by Remark 5.4 (a) that the *r-height* admits the following *geometric interpretation*:

By following Varchenko's algorithm (cf. Subsection 8.2 of [38]), one realizes that the principal line L of $\mathcal{N}(\phi)$ is a supporting line also for the Newton polyhedron of ϕ^a , which intersects $\mathcal{N}(\phi^a)$ in a compact face, either in a single vertex, or a compact edge. I.e., the intersection contains at least one and at most two vertices of $\mathcal{N}(\phi^a)$, and we choose (A_{l_0-1}, B_{l_0-1}) as the one with smallest second coordinate. Then l_0 is the smallest index l such that γ_l has a slope smaller than the slope of L , i.e., $a_{l_0-1} \leq m < a_{l_0}$. We may thus consider the *augmented* Newton polyhedron $\mathcal{N}^r(\phi^a)$ of ϕ^a , which is the convex hull of the union of $\mathcal{N}(\phi^a)$ with the half-line $L^+ \subset L$ with right endpoint (A_{l_0-1}, B_{l_0-1}) . Then $h^r(\phi) + 1$ is the second coordinate of the point at which the line $\Delta^{(m)}$ intersects the boundary of $\mathcal{N}^r(\phi^a)$.

The main result from [41], [42] then reads as follows.

Theorem 5.5. *Let $\phi \neq 0$ be real analytic, and assume that there is no linear coordinate system adapted to ϕ . Then there exists a neighborhood $U \subset S$ of $x^0 = 0$ such that for every non-negative density $\rho \in C_0^\infty(U)$, the Fourier restriction estimate (5.1) holds true for every $p \geq 1$ such that $p' \geq p'_c := 2h^r(\phi) + 2$.*

Remarks 5.6. (a) *An application of Greenleaf's result would imply, at best, that the condition $p' \geq 2h(\phi) + 2$ is sufficient for (5.1) to hold, which is a strictly stronger condition than $p' \geq p'_c$.*
 (b) *A. Seeger recently informed me that in a preprint, which regretfully had remained unpublished, Schulz [58] had already observed this kind of phenomenon for particular examples of surfaces of revolution.*
 (c) *It can be shown that the number m is well-defined, i.e., it does not depend on the chosen linearly adapted coordinate system x .*

Example 5.7.

$$\phi(x_1, x_2) := (x_2 - x_1^m)^n, \quad n, m \geq 2.$$

The coordinates (x_1, x_2) are not adapted. Adapted coordinates are $y_1 := x_1, y_2 := x_2 - x_1^m$, in which ϕ is given by

$$\phi^a(y_1, y_2) = y_2^n.$$

Here

$$\begin{aligned} \kappa_1 &= \frac{1}{mn}, \quad \kappa_2 = \frac{1}{n}, \\ d := d(\phi) &= \frac{1}{\kappa_1 + \kappa_2} = \frac{nm}{m+1} < n, \end{aligned}$$

and

$$p'_c = \begin{cases} 2d + 2, & \text{if } n \leq m + 1, \\ 2n, & \text{if } n > m + 1. \end{cases}$$

On the other hand, $h := h(\phi) = n$, so that $2h + 2 = 2n + 2 > p'_c$.

An analogous theorem holds true even for smooth, finite type functions ϕ , under an additional *Condition (R)* which, roughly speaking, requires that whenever the Newton diagram suggests that a root with leading term given by the principal root jet $\psi(x_1)$ should have multiplicity B , then indeed such a root of multiplicity B does exist (this is a condition on the behavior of flat terms). Condition (R) is always satisfied when ϕ is real-analytic.

For example, Condition (R) would hold true for

$$\phi_g(x_1, x_2) = (x_2 - x_1^2 - f(x_1))^2,$$

for every flat smooth function $f(x_1)$ (i.e., $f^{(j)}(0) = 0$ for every $j \in \mathbb{N}$). On the other hand, (R) is not satisfied for

$$\phi_b(x_1, x_2) := (x_2 - x_1^2)^2 + f(x_1),$$

unless f vanishes identically.

I also like to mention that there is a more invariant description of the notion of r -height (cf. Proposition 1.9 in [41]), somewhat in the spirit of Varchenko's definition of height, but I refrain from stating it here since this would require the introduction of further, somewhat technical notions.

5.1. Necessity of the condition $p' \geq 2h^r(\phi) + 2$. In order to better understand the meaning of the notion of r -height, let me present the proof of the necessity of the condition $p' \geq 2h^r(\phi) + 2$ for the validity of the Fourier restriction estimate (5.1) when $\rho(x^0) \neq 0$. The proof will be based on a modified Knapp-type argument.

Let γ_l be any edge of $\mathcal{N}(\phi^a)$ with $a_l > m$, and choose the weight κ^l such that γ_l lies on the line L_l given by $\kappa_1^l t_1 + \kappa_2^l t_2 = 1$. Consider the region

$$D_\varepsilon^a := \{y \in \mathbb{R}^2 : |y_1| \leq \varepsilon^{\kappa_1^l}, |y_2| \leq \varepsilon^{\kappa_2^l}\}, \quad \varepsilon > 0,$$

in adapted coordinates y . In the original coordinates x , it corresponds to

$$D_\varepsilon := \{x \in \mathbb{R}^2 : |x_1| \leq \varepsilon^{\kappa_1^l}, |x_2 - \psi(x_1)| \leq \varepsilon^{\kappa_2^l}\}.$$

Assume that ε is sufficiently small. Since

$$\phi^a(\varepsilon^{\kappa_1^l} y_1, \varepsilon^{\kappa_2^l} y_2) = \varepsilon \left(\phi_{\kappa^l}^a(y_1, y_2) + O(\varepsilon^\delta) \right)$$

for some $\delta > 0$, we have that $|\phi^a(y)| \leq C\varepsilon$ for every $y \in D_\varepsilon^a$, i.e.,

$$(5.7) \quad |\phi(x)| \leq C\varepsilon \quad \text{for every } x \in D_\varepsilon.$$

Moreover, for $x \in D_\varepsilon$,

$$|x_2| \leq \varepsilon^{\kappa_2^l} + |\psi(x_1)| \lesssim \varepsilon^{\kappa_2^l} + \varepsilon^{m\kappa_1^l}.$$

Since $m \leq a_l = \kappa_2^l / \kappa_1^l$, we find that

$$|x_2| \lesssim \varepsilon^{m\kappa_1^l},$$

so that we may assume that D_ε is contained in the box where $|x_1| \leq \varepsilon^{\kappa_1^l}, |x_2| \leq \varepsilon^{m\kappa_1^l}$. Choose f_ε such that

$$\widehat{f}_\varepsilon(x_1, x_2, x_3) = \chi_0\left(\frac{x_1}{\varepsilon^{\kappa_1^l}}\right) \chi_0\left(\frac{x_2}{\varepsilon^{m\kappa_1^l}}\right) \chi_0\left(\frac{x_3}{\varepsilon}\right).$$

Then by (5.7) we see that $\widehat{f}_\varepsilon(x_1, x_2, \phi(x_1, x_2)) \geq 1$ on D_ε , hence, if $\rho(0) \neq 0$, then

$$\left(\int_S |\widehat{f}_\varepsilon|^2 \rho d\sigma \right)^{1/2} \geq |D_\varepsilon|^{1/2} = \varepsilon^{(\kappa_1^l + \kappa_2^l)/2}.$$

Since $\|f_\varepsilon\|_p \simeq \varepsilon^{((1+m)\kappa_1^l + 1)/p'}$, we find that the restriction estimate can hold true only if

$$p' \geq 2 \frac{(1+m)\kappa_1^l + 1}{\kappa_1^l + \kappa_2^l} = 2h_l + 2,$$

where we recall that $h_l = (1 + m\kappa_1^l - \kappa_2^l) / (\kappa_1^l + \kappa_2^l)$.

Notice that the argument still works if we replace the previous line L_l by the line L associated to the weight κ , and $\phi_{\kappa_l}^a$ by ϕ_{κ}^a . Since here $m\kappa_1 = \kappa_2$, this leads to the condition $p' \geq 2d + 2$, so that altogether necessarily

$$p' \geq 2 \max(d, \max_{l: a^l > m} h_l) + 2 = 2h^r(\phi) + 2.$$

Q.E.D.

5.2. Sufficiency of the condition $p' \geq 2h^r(\phi) + 2$: I. Some key steps in the proof when $h_{\text{lin}}(\phi) \geq 2$. Let us assume that we are working in linearly adapted coordinates x , so that $d := d(\phi) = h_{\text{lin}} := h_{\text{lin}}(\phi)$.

In the preceding discussions of problems A and B, it had been natural to distinguish between the cases where $h := h(\phi) < 2$ and where $h \geq 2$, since in the latter case, in many situations a reduction to a one-dimensional situation had been possible by means of the van der Corput type Lemma 3.2. For similar reasons, in the discussion of Problem C it appears natural to distinguish between the cases where $d < 2$ and where $d \geq 2$. In addition, when $d \geq 2$, it turns out that the case where $d \geq 5$ can be handled in a somewhat simplified way compared to the case where $2 \leq d < 5$.

We shall therefore assume in this section that $d \geq 5$. In a *first step*, we can again localize to the narrow κ -homogeneous subdomain (3.7) of the curve $x_2 = b_1 x_1^m$ given by

$$|x_2 - c x_1^m| \leq \varepsilon x_1^m,$$

by means of a cut-off function ρ_1 . Indeed, the technique of proof that we used in the case of adapted coordinates can essentially be carried over to the domain complementary to (3.7) without major new ideas, since one can show that the Fourier transforms of the corresponding dyadic pieces μ_k of the measure μ satisfy estimates of the form

$$|\widehat{\mu_k}(\xi)| \leq C 2^{-k|\kappa|} (1 + 2^{-k} |\xi_3|)^{-1/d}.$$

Notice also that $h^r := h^r(\phi) \geq d$.

Let us again assume for instance that the principal face of the Newton polyhedron of ϕ^a is a compact edge. Using the same notation as in Section 3, we choose again $\lambda > l_0$ so that the edge $\gamma_\lambda = [(A_{\lambda-1}, B_{\lambda-1}), (A_\lambda, B_\lambda)]$ is the principal face $\pi(\phi^a)$ of the Newton polyhedron of ϕ^a .

In a *second step*, we again narrow down the domain (3.7) to the neighborhood $D_{\text{pr}} := D_\lambda$ of the principal root jet given by (3.11), where

$$|x_2 - \psi(x_1)| \leq N_\lambda x_1^{a_\lambda},$$

again by decomposing the difference set of the domains (3.7) and (3.11) into the domains

$$D_l := \{(x_1, x_2) : \varepsilon_l x_1^{a_l} < |x_2 - \psi(x_1)| \leq N_l x_1^{a_l}\}, \quad l = l_0, \dots, \lambda - 1,$$

and the intermediate domains

$$E_l := \{(x_1, x_2) : N_{l+1} x_1^{a_{l+1}} < |x_2 - \psi(x_1)| \leq \varepsilon_l x_1^{a_l}\}, \quad l = l_0, \dots, \lambda - 1,$$

as well as $E_{l_0-1} := \{(x_1, x_2) : N_{l_0} x_1^{a_{l_0}} < |x_2 - \psi(x_1)| \leq \varepsilon_1 x_1^m\}$.

Contribution by the domains E_l . Let us denote by μ_{E_l} the contribution of the transition domains E_l to the measure μ . We then decompose μ_{E_l} bi-dyadically w.r. to the adapted coordinates y as

$$\mu_{E_l} = \sum_{j,k} \mu_{j,k},$$

so that $\mu_{j,k}$ is supported where $y_1 = x_1 \sim 2^{-j}$ and $y_2 = x_2 - \psi(x_1) \sim 2^{-k}$. Observe that this a “curved rectangle” in our original coordinates x . In a similar way as in the case of adapted coordinates, we would like to localize to these curved rectangles by means of Littlewood-Paley theory in order to reduce to uniform restriction estimates for the family of measure $\mu_{j,k}$, i.e.,

$$(5.8) \quad \int_S |\widehat{f}|^2 d\mu_{j,k} \leq C \|f\|_{L^p(\mathbb{R}^3)}^2,$$

for $p \leq p_c$. Clearly, because of the non-linearity $\psi(x_1)$, this is not possible by means of Littlewood-Paley techniques in the variables x_1 and x_2 , but it turns out we can use the variables x_1 and x_3 to accomplish this.

Indeed, one can show that

$$\phi^a(y) = c_l y_1^{A_l} y_2^{B_l} (1 + \text{small error}) \quad \text{on } E_l^a,$$

which in return implies that on the domains E_l respectively E_l^a (recall that E_l^a represents E_l in the adapted coordinates y) the conditions $y_1 \sim 2^{-j}, y_2 \sim 2^{-k}$ are equivalent to the conditions

$$x_1 \sim 2^{-j} \quad \text{and} \quad \phi(x) \sim 2^{-(A_l j + B_l k)}$$

(cf. Lemma 6.1 in [41]).

Working in the coordinates y , after re-scaling of the measures $\mu_{j,k}$ to get normalized measures $\nu_{j,k}$ supported on a surface $S_{j,k}$ where $y_1 \sim 1 \sim y_2$, by means of the formula above one eventually finds that $S_{j,k}$ is a small perturbation of the limiting surface

$$S_\infty := \{(y_1, y_1^m \omega(0), c y_1^{A_l} y_2^{B_l}) : y_1 \sim 1 \sim y_2\},$$

But $|\partial(c y_1^{A_l} y_2^{B_l})/\partial y_2| \sim 1$, since $B_l \geq 1$, which shows that S_∞ , and hence also $S_{j,k}$, is a smooth hypersurface with one non-vanishing principal curvature (with respect to y_1) of size ~ 1 . This implies that

$$|\widehat{\nu_{j,k}}(\xi)| \leq C(1 + |\xi|)^{-1/2},$$

uniformly in j and k . Applying Greenleaf’s restriction theorem to these measures, and scaling these estimates back, we eventually arrive (in a not completely trivial way) at the estimates (5.8). It is important to observe here that Greenleaf’s results implies restriction theorems for $p' \geq 2(1 + 2) = 6$, which is sufficient for our purposes, since $p'_c \geq 2d + 2$, where $d \geq 2$.

Contribution by the domains D_l . Let us next turn to the domains D_l . After dyadic decomposition of the domain D_l in the adapted coordinates y by means of the κ^l -dilations and suitable re-scaling, the re-scaled measure ν_k corresponding to the measures μ_k turns out to be of the form

$$\langle \nu_k, f \rangle := \int f(y_1, 2^{(m\kappa_1^l - \kappa_2^l)k} y_2 + y_1^m \omega(2^{-\kappa_1^l k} y_1), \phi^k(y)) \tilde{\eta}(y) dy,$$

and by means of a finite partition of unity, we may assume that the amplitude $\tilde{\eta}$ is supported in a sufficiently thin set $U(c_0)$, on which

$$y_1 \sim 1 \quad \text{and} \quad |y_2 - c_0 y_1^{a_l}| \leq \varepsilon y_1^{a_l}.$$

This measure ν_k is supported in a variety S_k which in the limit as $k \rightarrow \infty$ tends to the variety

$$S_\infty := \{g_\infty(y_1, y_2) := (y_1, \omega(0)y_1^m, \phi_{\kappa^l}^a(y)) : (y_1, y_2) \in U(c_0)\},$$

since $m\kappa_1^l - \kappa_2^l < a_l \kappa_1^l - \kappa_2^l = 0$ and since ϕ^k tends to $\phi_{\kappa^l}^a$. Here, c_0 is fixed with $|c_0| \leq N_l$.

Again, we have to prove uniform restriction estimates for the family of measures ν_k . Depending on c_0 , different cases may arise.

1. Case. $\partial_2 \phi_{\kappa^l}^a(1, c_0) \neq 0$. Then we may use $z_2 := \phi_{\kappa^l}^a(y_1, y_2)$ in place of y_2 as a new coordinate for S_∞ (which thus is a hypersurface), and since $y_1 \sim 1$ on $U(c_0)$, we find that S_∞ , hence also S_k , is a hypersurface with *one non-vanishing principal curvature*. Then we may essentially argue as for the domains E_l .

2. Case. $\partial_2 \phi_{\kappa^l}^a(1, c_0) = 0$, but $\partial_1 \phi_{\kappa^l}^a(1, c_0) \neq 0$. In this case, since $\phi_{\kappa^l}^a$ is a κ^l -homogenous polynomial, by Euler's homogeneity relation we have also $\phi_{\kappa^l}^a(1, c_0) \neq 0$. One can then show that one can fibre the variety S_∞ into the family of curves

$$\gamma_c(y_1) := g_\infty(y_1, c y_1^{a_l}) = (y_1, \omega(0)y_1^m, \phi_{\kappa^l}^a(y_1, c y_1^{a_l})),$$

for c sufficiently close to c_0 , and one finds that the curve $\gamma_{c_0}(y_1) = (y_1, \omega(0)y_1^m, b_0 y_1^{1/\kappa_1^l})$ has *non-vanishing torsion*, since $b_0 \neq 0$. The same applies then to the curves γ_c , and for k sufficiently large, we do obtain the analogous results for the varieties S_k .

This allows to decompose the measure $d\nu_k$ as a direct integral of measures $d\Gamma_c$ supported on curves γ_c^l with non-vanishing torsion.

We may thus apply Drury's Fourier restriction theorem for curves with non-vanishing torsion (cf. Theorem 2 in [23] and [5], [20]) to the measures $d\Gamma_c$ and obtain uniform estimates

$$\left(\int |\hat{f}|^2 d\Gamma_c \right)^{\frac{1}{2}} \leq C_p \|f\|_{L^p(\mathbb{R}^3)},$$

when $p' > 7$ and $2 \leq p'/6$. Since we assume here that $p'_c \geq 2(d+1) > 2(5+1) = 12$, these estimates, after re-scaling to the measures μ_k , yield the desired restriction estimates for the contributions by the domains D_l .

Notice that it is here that we need the condition $d = h_{\text{lin}} > 5$.

3. Case. $\partial_2 \phi_{\kappa^l}^a(1, c_0) = 0$ and $\partial_1 \phi_{\kappa^l}^a(1, c_0) = 0$. Then Euler's homogeneity relation implies that also $\phi_{\kappa^l}^a(1, c_0) = 0$, so that $\phi_{\kappa^l}^a$ has a real root of multiplicity $B \geq 2$ at $(1, c_0)$, and one finds that

$$(5.9) \quad \phi_{\kappa^l}^a(y_1, y_2) = y_2^{B_l}(y_2 - c_0 y_1^{a_l})^B Q(y_1, y_2),$$

where Q is a κ^l -homogenous smooth function such that $Q(1, c_0) \neq 0$ and $Q(1, 0) \neq 0$. One can also prove that $B < d/2$.

We can then essentially follow the Stein-Tomas method for proving L^p - L^2 -restriction estimates. We localize to frequencies of size $\Lambda > 1$ by putting

$$\widehat{\nu_k^\Lambda}(\xi) := \chi_1\left(\frac{\xi}{\Lambda}\right) \widehat{\nu_k}(\xi),$$

where χ_1 is a smooth bump function supported where $|\xi| \sim 1$. We claim that the measures ν_k^Λ satisfy the following estimates, uniformly in $k \geq k_0$, provided k_0 is sufficiently large and ε' sufficiently small:

$$(5.10) \quad \|\widehat{\nu_k^\Lambda}\|_\infty \leq C\Lambda^{-1/B};$$

$$(5.11) \quad \|\nu_k^\Lambda\|_\infty \leq C\Lambda^{2-1/B}.$$

Indeed,

$$\widehat{\nu_k^\Lambda}(\xi) = \chi_1\left(\frac{\xi}{\Lambda}\right) \int e^{-i\left[\xi_1 y_1 + \xi_2 \left(2^{(m\kappa_1^l - \kappa_2^l)k} y_2 + y_1^m \omega(2^{-\kappa_1^l k} y_1)\right) + \xi_3 \phi_k(y)\right]} \tilde{\eta}(y) dy,$$

which, in the limit as $k \rightarrow \infty$, simplifies as

$$\widehat{\nu_\infty^\Lambda}(\xi) = \chi_1\left(\frac{\xi}{\Lambda}\right) \int e^{-i[\xi_1 y_1 + \xi_2 \omega(0) y_1^m + \xi_3 \phi_{\kappa^l}^a(y)]} \tilde{\eta}(y) dy.$$

Now, if $|\xi_3| \geq c|(\xi_1, \xi_2)|$, then an application of van der Corput's lemma to the integration in y_2 yields $|\widehat{\nu_\infty^\Lambda}(\xi)| \lesssim |\xi_3|^{-1/B}$ (cf. (5.9)), and if $|\xi_3| \ll |(\xi_1, \xi_2)|$, we may apply van der Corput's lemma to the y_1 -integration and obtain $|\widehat{\nu_\infty^\Lambda}(\xi)| \lesssim |(\xi_1, \xi_2)|^{-1/2}$. Since $B \geq 2$, and because van der Corput's estimates are stable under small perturbations, we thus obtain (5.10).

In order to verify (5.11), observe that $\nu_\infty^\Lambda(x_1, x_2, x_3)$ is given by

$$\Lambda^3 \int (\mathcal{F}^{-1} \chi_1)(\Lambda(x_1 - y_1), \Lambda(x_2 - \omega(0)y_1^m), \Lambda(x_3 - \phi_{\kappa^l}^a(y_1, y_2))) \tilde{\eta}(y) dy_1 dy_2,$$

hence, by a change of coordinates,

$$|\nu_\infty^\Lambda(x_1, x_2, x_3)| \leq \Lambda^2 \int \rho(z_1) \rho(\Lambda(x_3 - \phi_{\kappa^l}^a(x_1 - \frac{z_1}{\Lambda}, y_2))) \eta_1(x_1 - \frac{z_1}{\Lambda}, y_2) dz_1 dy_2,$$

where ρ and η_1 are suitable, non-negative Schwartz functions, and η_1 localizes again to $U(c_0)$. However, since $|\partial_2^B \phi_{\kappa^l}^a(y_1, y_2)| \simeq 1$ on the domain of integration, classical sub-level estimates, originating in work by van der Corput [16] (see also [1], and [13],[26]), essentially imply that the integral with respect to y_2 can be estimated by $O(\Lambda^{-1/B})$, uniformly in y_1 and Λ .

Interpolating the estimates (5.10) and (5.11), and applying the standard Stein-Tomas argument (see, for instance, [34]), it is easily seen that we can sum the corresponding estimates over all dyadic $\Lambda \gg 1$, and we obtain the L^p - L^2 restriction estimate

$$\left(\int |\widehat{f}|^2 d\nu_k \right)^{1/2} \leq C_p \|f\|_{L^p}$$

whenever $p' > 4B$, uniformly in k , for k sufficiently large. Since $B < d/2$, we have $p'_c \geq 2d + 2 > 4B$, so that the range $p > 4B$ does include the critical value $p = p_c$.

Scaling back to the measures μ_k , we find that also the original measures μ_k satisfy a uniform restriction estimate

$$\left(\int |\widehat{f}|^2 d\mu_k \right)^{1/2} \leq C_p \|f\|_{L^p},$$

where C_p does not depend on k , provided $p' \geq 2h_l + 2$. However, this applies to p_c , since $h^r(\phi) \geq h_l$.

Finally, observe that we can achieve our dyadic decomposition into the measures μ_k by means of a dyadic decomposition in the variable x_1 , so that these uniform estimates allow to sum over all k by means of Littlewood-Paley theory applied to variable x_1 !

What remains to be understood is the contribution by the domain $D_{\text{pr}} = D_\lambda$ given by

$$|x_2 - \psi(x_1)| \leq N_\lambda x_1^{a_\lambda}.$$

In this domain, the upper bound $B < d/2$ for the multiplicity B of real roots will in general no longer be valid, as examples show, not even the weaker condition $B < h^r(\phi)/2$, which would still suffice for the previous argument.

In order to resolve this problem, we apply again a *further domain decomposition* by means of a stopping time argument.

Notice first that if we are to proceed as for the case $l < \lambda$, a major problem arises only in Case 3 where $\nabla \phi_{\text{pr}}^a(1, c_0) = 0$; in all other cases we can essentially argue as before.

We therefore devise the *stopping time argument* essentially as follows:

We put $\phi^{(1)} := \phi^a$. If Case 3 does not appear for any choice of c_0 , then we stop our algorithm with $\phi^{(1)}$, and are done. Otherwise, if Case 3 applies to c_0 , so that $c_0 y_1^{a_\lambda}$ is a root of $\phi_{\kappa\lambda}^a$, then we define new coordinates z in place of y by putting

$$z_1 := x_1 \quad \text{and} \quad z_2 := x_2 - \psi(x_1) - c_0 x_1^{a_\lambda},$$

and express ϕ by $\phi^{(2)}$ in the coordinates z . Again, if Case 3 does not appear (for $\phi^{(2)}$ in place of $\phi^{(1)}$) in the corresponding z -domain, we stop our algorithm. We also stop the algorithm when no further fine splitting of roots does occur, i.e., when there is a root with leading term $\psi(x_1) + c_0 x_1^{a_\lambda}$ of ϕ which has the same multiplicity as the trivial root $z_2 = 0$ of the principal part of $\phi^{(2)}$. Otherwise, we continue in an analogous way.

This algorithm will stop after a finite number of steps, and eventually leads to a further domain decomposition of D_{pr} into “homogeneous” domains $D_{(l)}$ and transition

domains $E_{(l)}$, which can eventually be treated by methods similar to those applied for the domains E_l and D_l .

5.3. Sufficiency of the condition $p' \geq 2h^r(\phi) + 2$: II. Brief sketch of some ideas of the proof when $h_{\text{lin}}(\phi) < 2$. This case turns out to be by far more difficult than the case where $h_{\text{lin}}(\phi) \geq 5$, and its discussion requires numerous further methods and ideas which I can sketch only very briefly.

A first observation is that $h^r(\phi) = d$ when $d = h_{\text{lin}}(\phi) < 2$, so that

$$p'_c = 2d + 2.$$

The starting point of our analysis is the following local normal form of our function ϕ . It is closely related to the classification of singularities in [2] and [24].

Theorem 5.8. *If $h_{\text{lin}}(\phi) < 2$, then locally ϕ is of the form*

$$(5.12) \quad \phi(x_1, x_2) = b(x_1, x_2)(x_2 - \psi(x_1))^2 + b_0(x_1).$$

Here b, b_0 and ψ are smooth, and ψ is again the principal root jet, and either

- (a) $b(0, 0) \neq 0$, and either b_0 is flat (singularity of type A_∞), or of finite type n , i.e., $b_0(x_1) = x_1^n \beta(x_1)$, where $\beta(0) \neq 0$ (singularity of type A_{n-1});
- or
- (b) $b(0, 0) = 0$ and $b(x_1, x_2) = x_1 b_1(x_1, x_2) + x_2^2 b_2(x_2)$, with $b_1(0, 0) \neq 0$ (singularity of type D).

Let us just consider the case where ϕ is of finite type A_{n-1} (type D can be treated in a rather similar way).

In a *first step*, by making use of these normal forms in order to estimate certain two-dimensional oscillatory integrals that arise in estimating the Fourier transforms of surface carried measures, we can again reduce to the domain (3.7), where $|x_2 - cx_1^m| \leq \varepsilon x_1^m$.

In a *second step*, if κ denotes again the principal weight associated to the principal edge of $\mathcal{N}(\phi)$, we again perform a dyadic decomposition of the measure μ over the domain (3.7), and re-scale in a suitable way. This leads to a phase function

$$\phi(x, \delta) := b(\delta_1 x_1, \delta_2 x_2) \left(x_2 - x_1^m \omega(\delta_1 x_1) \right)^2 + \delta_0 x_1^n \beta(\delta_1 x_1),$$

where $\delta = (\delta_0, \delta_1, \delta_2) = (2^{-(n\kappa_1-1)k}, 2^{-\kappa_1 k}, 2^{-\kappa_2 k})$ are small parameters which tend to 0 as k tends to infinity, and where $b(\delta_1 x_1, \delta_2 x_2) \sim b(0, 0) \neq 0$ and $\beta(0) \neq 0$.

What we then need to prove is the following

Proposition 5.9. *Given any point $v = (v_1, v_2)$ such that $v_1 \sim 1$ and $v_2 = v_1^m \omega(0)$, there exists a neighborhood V of v in $(\mathbb{R}_+)^2$ such that for every cut-off function $\eta \in \mathcal{D}(V)$, the measure ν_δ given by*

$$\langle \nu_\delta, f \rangle := \int f(x, \phi(x, \delta)) \eta(x_1, x_2) dx$$

satisfies a restriction estimate

$$\left(\int |\widehat{f}|^2 d\nu_\delta \right)^{1/2} \leq C_{p,\eta} \|f\|_{L^p(\mathbb{R}^3)},$$

whenever $p' \geq 2d + 2$, provided δ is sufficiently small.

In order to prove this proposition, we again perform a dyadic decomposition, this time with respect to the x_3 -variable. By means of Littlewood-Paley theory, it then turns out that it is sufficient to prove uniform restriction estimates for the following family of measures

$$\langle \nu_{\delta,j}, f \rangle := \int f(x, \phi(x, \delta)) \chi(2^{2j}\phi(x, \delta)) \eta(x_1, x_2) dx,$$

of the form

$$\left(\int |\widehat{f}|^2 d\nu_{\delta,j} \right)^{1/2} \leq C_{p,\eta} \|f\|_{L^p(\mathbb{R}^3)}.$$

Here, $\chi(x_3)$ is supported on a set where $|x_3| \sim 1$. If $2^{2j}\delta_0 \ll 1$, then it turns out that this localization means in fact again a localization to a curved rectangle where

$$|x_1 - v_1| < \varepsilon \quad \text{and} \quad |x_2 - x_1^m \omega(\delta_1 x_1)| \sim 2^{-j},$$

but in other cases, it has another meaning.

It turns out that these estimates require a *refined spectral decomposition* of the measures $\nu_{\delta,j}$, namely a dyadic decomposition in every dual coordinate ξ_1, ξ_2, ξ_3 . Slightly cheating, for every triple $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ of dyadic numbers $\lambda_i = 2^{-k_i}$, we therefore define the functions ν_j^Λ by

$$\widehat{\nu_j^\Lambda}(\xi) = \chi_1\left(\frac{\xi_1}{\lambda_1}\right) \chi_1\left(\frac{\xi_2}{\lambda_2}\right) \chi_1\left(\frac{\xi_3}{\lambda_3}\right) \widehat{\nu_{\delta,j}}(\xi),$$

and choose $\chi_1(s)$ supported where $|s| \sim 1$ so that $\nu_{\delta,j} = \sum_\Lambda \nu_j^\Lambda$, where summation is essentially over all these dyadic triples Λ . We have here suppressed the dependency of ν_j^Λ on the small parameters δ . Note that $|\xi_i| \sim \lambda_i$ on the support of $\widehat{\nu_j^\Lambda}$.

For every fixed Λ , we then essentially follow again the Stein-Tomas approach, by estimating $\|\widehat{\nu_j^\Lambda}\|_\infty$ and $\|\nu_j^\Lambda\|_\infty$.

This requires the distinction of various cases, depending on the relative sizes of λ_1, λ_2 and λ_3 . In the end, it turns out that the most difficult case is where $\lambda_1 \sim \lambda_2 \sim \lambda_3$ and $2^{2j}\delta_0 \sim 1$, and the main result to be proven in this case is the following.

Proposition 5.10. *Let ϕ be of type A_{n-1} , with $m = 2$ and finite $n \geq 5$. Then*

$$(5.13) \quad \sum_{2 \leq \lambda_1 \sim \lambda_2 \sim \lambda_3 \leq 2^{6j}} \int_S |\widehat{f}|^2 d\nu_j^\Lambda \leq C 2^{\frac{1}{7}j} \|f\|_{L^{14/11}(\mathbb{R}^3)}^2,$$

for all $j \in \mathbb{N}$ sufficiently big, say $j \geq j_0$, where the constant C does neither depend on δ , nor on j .

The proof of this result requires yet further refinements. Indeed, the Fourier transform of ν_j^Λ is an oscillatory integral with complete phase of the form

$$\begin{aligned}\Phi(y; \delta, j, \xi) &= \xi_1 y_1 + \xi_2 y_1^2 \omega(\delta_1 y_1) + \xi_3 \sigma y_1^n \beta(\delta_1 y_1) \\ &\quad + 2^{-j} \xi_2 y_2 + \xi_3 b^\sharp(y, \delta, j) y_2^2,\end{aligned}$$

where $\sigma := 2^{2j} \delta_0 \sim 1$ and $|b^\sharp(y, \delta, j)| \sim 1$. Notice that if $|\xi_1| \sim |\xi_2| \sim |\xi_3|$, then ϕ may have degenerate critical points, with non-vanishing third derivatives, with respect to the variable y_1 , so that we encounter oscillatory integrals of “Airy type”.

This case requires a *further dyadic frequency decomposition with respect to the distance to certain “Airy cones.”* For the quite comprehensive details, I refer to [41].

The case where $2 \leq d < 5$ turns out to be the most delicate one, since on the one hand, we can no longer apply Drury’s restriction theorem, which necessitates an even more refined domain decomposition, and furthermore, somewhat subtle interpolation arguments are needed in many situations in order to handle the endpoint $p = p_c$ (cf. [42]).

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